Solving Quadratic Equations

Formal method for solving quadratic Equations (and higher order equations). Suppose $f(x)=0$ In General , to solve a linear equation, we solve the equation by isolating the variable onto one side of the equation. On the contrary, if $f(x)$ is higher than linear order, we begin by collecting all terms to one side and putting the equation in the form $f(x)=0$.

So, suppose $f(x)=0$ and $f(x)$ is quadratic. Then, in order to solve a quadratic Equation (i.e. an equation of the form f(x)=0, where f (x) is a quadratic polynomial), we generally have three methods:

Method 1 (Factoring Method): The first method is to solve *f(x)=0* by factoring. This method is the best method and also usually the fastest method provided the student has much practice with this factoring. Unfortunately, not every quadratic polynomial can be factored over the integers. Thus, the factoring method has a very significant drawback!

Method 2 (Completing the Square): We rarely use this property because it is tedious. However, its primary purpose of existence is to derive the quadratic formula, for which every quadratic equation can be solved by using the quadratic formula (and as a result completing the square). As a result, we must understand this process, how to solve quadratics using this method, and how to derive the quadratic formula from this method of completing the square.

Method 3 (The Quadratic Formula): We will use completing the square (i.e. method 2 above) to find a formula for the x-value(s) of $f(x)=0$. The benefit of this method is that it always works!

Special Case: If $b=0$ in the quadratic equation $ax^2+bx+c=0$ (i.e. $f(x)$ missing the middle term) , then the square root property may be a shortcut.

Example: Solve: $x^2=4$

Solution: By taking the (plus or minus) square root of both sides, we clearly see that $x=\{-2,2\}$.

QED

Example: Solve $(x+2)^2=9$

Solution: By taking the (plus or minus) square root of both sides, we obtain:

$x+2=3$ or $x+2=-3$. Thus, $x=\{1,-5\}$

QED

Zero-Factor Property:

If $ab=0$, then $a=0$ or $b=0$

Example: Solve by factoring: $x^2-x=6$

Solution: The first step to solving this equation is to get zero on one side (i.e. get into the form $f(x)=0$. So, by subtracting 6 from both sides we obtain $x^2-x-6=0$. So,

$x^2-x-6=0$

$(x-3)(x+2)=0$ Factor

$x-3=0$ or $x+2=0$ Zero-Factor Property

$x=3$ *or* $x=-2$ Solve each equation

$x=\{-2,3\}$ Express solution in set notation

QED

Example: Solve by completing the square: $-x^2-2x+8=0$

Solution: $-x^2-2x+8=0$

$-x^2-2x+8=0$

$-(x^2+2x\quad)=-8$ Factor out the negative,leave space to complete the square, and bring the 8 over

$-(x^2+2x+1)=-8-1$ Complete the square by "adding the sqare of half the middle term" to both sides

$-(x+1)^2=-9$ Re-arrange terms

$(x+1)^2=9$

$x+1=-3$ or $x+1=3$ Square Root Property

$x=-4$ *or* $x=2$ Solve each linear equation

$x=\{-4,2\}$ Express answer in set notation

QED

Theorem (The Quadratic Formula): If $f(x) = ax^2 + bx + c = 0$ $(a \neq 0)$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof:

Suppose $f(x) = ax^2 + bx + c = 0$ $(a \neq 0)$. Lets complete the square to solve the problem. Thus,

$f(x) = ax^2 + bx + c = 0$ $(a \neq 0)$	
$f(x) = a\left(x^2 + \frac{b}{a}x \quad\right) = -c$	Factor out the leading coefficicident
$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = -c + \frac{b^2}{4a}$	Complete the square
$= a\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a} + \frac{b^2}{4a}$	Rewrite and get LCD for Righ hand side of the equation
$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$	Rearrange the terms
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Multiply both sides by $\frac{1}{a}$ (recall $a \neq 0$)
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Square Root Property
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Simplify
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Additon Property of Eaquality
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Simplify

QED

Example: Solve by using the quadratic formula: $-x^2 - 2x = -8$.

Solution:

$$-x^2 - 2x = -8$$
$$-x^2 - 2x + 8 = 0 \qquad \text{Get zero on one side (i.e set f}\left(\text{x}\right) = 0\text{)}$$

From the quadratic formula, we have $a = -1, b = -2,$ and $c = 8$. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4\left(-1\right)\left(8\right)}}{2\left(-1\right)} = \frac{2 \pm \sqrt{4 + 32}}{-2} = \frac{2 \pm \sqrt{36}}{-2}$$
$$= \frac{2 \pm 6}{-2} = \left\{\frac{2+6}{-2}, \frac{2-6}{-2}\right\} = \left\{-4, 2\right\}.$$

QED

Example: Solve by using the method of your choice: $x^2 - 10x + 23 = 0$

Solution: Since 23 is prime, you can see that the polynomial does not factor. So we should go directly to the quadratic formula with $a = 1, b = -10,$ and $c = 23$. Proceeding with this, we obtain:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 - 4(1)(23)}}{2}$$
$$= \frac{10 \pm \sqrt{100 - 92}}{2} = \frac{10 \pm \sqrt{8}}{2} = \frac{10 \pm 2\sqrt{2}}{2}$$
$$= 5 \pm \sqrt{2} = \left\{5 - \sqrt{2}, 5 + \sqrt{2}\right\}.$$

QED