Graphing Quadratic Equations (i.e. Parabolas)

Definition: A *quadratic Equation* is an equation of the form $y = ax^2 + bx + c$ where $a \neq 0$.

Note that in the a above definition, a quadratic equation is just a second degree polynomial. The graph of the quadratic equation is called a *parabola*. We will study how to graph such parabolas (i.e. the set of all points that satisfy the quadratic equation!)

The above form of the second-degree polynomial given in the definition is called the standard form. However, we can algebraically manipulate the standard form of the equation into a new form called *vertex form*. As a result, quadratic equations are generally expressed in one of two forms. Standard form and vertex form. One can convert back and forth easily. For example, to convert from vertex form to standard form, one would just simplify. On the contrary, converting from standard form to vertex form is not as easy. Recall, we must complete the square.

In short, there exists two forms of the quadratic equation:

1. Standard Form: $y = ax^2 + bx + c$

2. Vertex Form: $y = a\left(x-h\right)^2 + k$ [the vertex is at $\left(h,k\right)$]

As you will see, the vertex form is very convenient because we can obtain the vertex directly from the equation by understanding that the vertex form is just a transformation of the graph of $y = x^2$. Thus, you can see the transformation would include shifting right h units and up k units. Thus, for the vertex form, the vertex is at $\left(h,k\right)$.

To graph the parabola, we will need the vertex and a few points on one side of the vertex. Recall, we can get the points on the other side of the vertex by reflecting each point about the vertical line $x = h$ (i.e. the vertical line through the vertex).

Example:

Graph: $y = -\left(x-1\right)^2 + 3$.

Solution:

We can see the vertex is at $\left(1,3\right)$. Choose a few x-values to the left or right of x = 1, such as 2, 3, and 4. We obtain the points $\left(2,2\right),\left(3,-1\right),\left(4,-6\right)$. Reflecting these points through the vertical line $x = 1$, we also obtain the points $\left(0,2\right),\left(-1,-1\right),\left(-2,-6\right)$ on the graph. Connecting the points with a smooth curves gives the desired parabola.

Note to the instructor: Be sure to emphasize how this parabola can be obtained (and should be visualized before plotting points) by shifting $y = x^2$ one unit to the right, reflecting about the x-axis, and then 3 units up.

QED

Theorem: If $y = ax^2 + bx + c$, then the vertex of the parabola is at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Proof: We can complete the square to write f(x) as

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c && \text{rewrite} \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} && \text{complete the square} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \\
&= a\left(x - \left(\frac{-b}{2a}\right)\right)^2 + \left(c - \frac{b^2}{4a}\right) && \text{Express in Vertex Form} \\
&= a\left(x - \left(\frac{-b}{2a}\right)\right)^2 + f\left(\frac{-b}{2a}\right)
\end{aligned}
$$

Note to the instructor: You may want to show the student that $f\left(\frac{-b}{2a}\right) = c - \frac{b^2}{4a}$ and review the completing the square concept before presenting this proof with the theorem.

Example: Graph $y = 3x^2 - 6x + 3$

Solution: According to the above theorem, the vertex is at (h, k), where $h = \frac{-b}{2a} = \frac{-(-6)}{2 \cdot 3} = \frac{6}{6} = 1$ and $k = f\left(\frac{-b}{2a}\right) = f(1) = 0$. So, the vertex is $V(1,0)$. As done in the previous example, we can get points on both sides of the vertex by plotting a few points to the left *or* right of $x = 1$ and then reflecting.