Equations and Inequalities

Solving Linear Equations in One Variable and Simplifying Algebraic Expressions

Definition: An *algebraic expression* is an expression that includes basic operations such as addition, subtraction, multiplication, division (but not by zero), roots, or any other expression containing variables and numbers.

Remark: An *algebraic expression* contains no equal or inequality symbols. That is one way that we can differentiate an algebraic expression from an algebraic equation or inequality, which will be addressed later in this section.

Example: Some examples of algebraic expressions are given below:

a. $4(x+3)$

b. $\sqrt{x}+2$

c. $\dfrac{xy^2}{2z}$

d. $4[-3(x+3)+2(3x-1)]-3(2x+1)$

QED

Simplifying Algebraic Expressions

When we simplify algebraic expressions we need to work inside-out (i.e. simplify the inner most parentheses first and work your way out), while at the same time obeying the order of operations. Recall, the order of operations can be remembered by the acronym PEMDAS (**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally). Thus, we work parentheses first, followed by exponents, then multiplication and division, then addition and subtraction.

Multiplication and division come as one (since they are the same) and addition and subtraction as well. Also if there is ever a "Tie" (i.e. there exists two multiplications in the same expression), then we simplify left to right while obeying the parentheses and order of operations. Most importantly, parentheses trump the order of operations. That is if an expression contains parentheses, then we will simplify inside these parentheses before continuing with the order of operations.

Example: Simplify: $(2+6)\div 3$

Although we would normally calculate $6\div 3$ first as the order of operations demands (since multiplication/Division comes before addition/subtraction), the parentheses are making it clear that instead we are to simplify $2+6$ first. So, we obtain:

$$(2+6)\div 3 = 8\div 3 = \frac{8}{3}$$

Caution: Without the parentheses we would obtain $2+6\div 3 = 2+\left(6\div 3\right) = 2+2 = 4$

QED

Example: Simplify: $-2(x+1)+3x-(2x-3)$

$-2(x+1)+3x-(2x-3)$

$= -2x-2+3x-2x+3$ Distributive Property

$= \ -x+1$ combine like terms

QED

Example: Simplify: $2x+3-4(x+2)-7$

$2x+3-4(x+2)-7$

$= 2x+3-4x-8-7$ Distributive Property

$= \ -2x-12$ combine like terms

QED

Example: Simplify: $4\left[3(2y-5)+5y-(y+1)\right]$

$4\left[3(2y-5)+5y-(y+1)\right]$ Distributive Property on the inside first

$= 4[6y-15+5y-y-1]$ combine like terms working inside out

=4[10y-16] Distributive Property

=40y-64

QED

Solving Linear Equations in One Variable

Definition: A *linear Equation* is any equation that can be written in the form $ax+b=c$, where $a, b,$ and c are real numbers and $a \neq 0$. Sometimes a linear equation is called a first degree equation since the highest power on the variable is 1.

Addition Property of Equality:

For all real numbers a, b, and c, the equations $a=b$ and $a+c=b+c$ are equivalent.

(This property states that the same number may be added or subtracted to both sides of any equation).

Example:

Solve for x: $x+7=12$

By subtracting 7 from both sides, we obtain:

$x+7=12$

-7 -7 Addition Property of Equality with c = -7

$x=5$

Although a check is not required as is in radical equations, a check is recommended:

Check:

$x+7=12$

$5+7=12$ Substitute 5 for x

$12=12$ True statement

Thus, are solution is correct!

QED

In the previous example, we subtracted 7 from both sides (i.e. we added -7). After one gets comfortable with the addition property of equality, the student will realize that any time a number (or variable) moves over the equal sign, we change the sign of that number or variable. For instance in the previous example, we moved the +7 over to the right side of the equation to isolate the x and it became -7 on the right side of the equation. Combining -7 with the +12 that was already on the right side yields and answer of x = 5.

Shortcut: When applying the Addition Property of Equality to solve a linear equation, anytime a number or variable moves over the equal sign (in an effort to isolate the variable), we change the sign of that number or variable.

Example:

Solve for y: $y-5=9$

By adding 5 to both sides, we obtain:

$$\begin{array}{l} y-5=9 \\ \quad +5 \quad +5 \\ \hline y = \quad 14 \end{array}$$

Addition Property of equality with c = 5

Remark regarding shortcut: We move the -5 to the right side of the equation in an effort to isolate the variable y and the -5 on the left side becomes + 5 on the right side of the equation. So, the left side has y isolated and the right side becomes +5 plus the 9 that was already on the right side, yielding 14.

Check:

$y-5=9$

$14-5=9$ Substitute 14 for y

$9=9$ True Statement

Thus, the solution is correct!

QED

Multiplication Property of Equality:

For all real numbers a,b,c (With $c \neq 0$), the equations $a=b$ and $ac=bc$ are equivalent.

(This property states we may multiply or divide both sides of any equation by any non-zero real number).

Example:

Solve for x: $3x = 12$

By dividing both sides of the equation by 3 we obtain:

$3x = 12$

$x = 4$ Multiplication Property of Equality with $c = \frac{1}{3}$

Check:

$3x = 12$

$3(4) = 12$ Replace x by 4

12=12 True statement

Thus the solution is correct!

QED

Example: Solve for x: $\frac{2}{3}x = 6$

By multiplying both sides of the equation by the reciprocal of $\frac{2}{3}$ or $\frac{3}{2}$, we obtain:

$\frac{2}{3}x = 6$

$\frac{3}{2} \cdot \frac{2}{3}x = \frac{6}{1} \cdot \frac{3}{2}$ Multiplicative Property of Eqality with c=$\frac{3}{2}$

$1x = 9$

$x = 9$

Check: $\frac{2}{3} \cdot \frac{9}{1} = 6$

QED

Example: Solve for m: $-2m + 3 = 5$

$-2m+3=5$

$-2m=2$ Addition Property of Equality with c = -3

$m=-1$ Multiplication Property of Equality with c = -2

Check: $-2(-1)+3=5$

QED

Example: Solve for x: $3(x-2)+2x=-2(3x+1)$

$3(x-2)+2x=-2(3x+1)$

$3x-6+2x=-6x-2$ Distribute

$5x-6$ =-6x-2 combine like terms (i.e. simplify expressions on each side of the equation)

11x=4 Additive Property of Equality

x=$\frac{4}{11}$ Multiplicative Property of Equality

Check:

$3(x-2)+2x=-2(3x+1)$

$3\left(\frac{4}{11}-2\right)+2\left(\frac{4}{11}\right)=-2\left(3\left(\frac{4}{11}\right)+1\right)$ Substitute $\frac{4}{11}$ for x

$3\left(\frac{-18}{11}\right)+\frac{8}{11}$ $= -2\left(\frac{12}{11}+1\right)$ Simplify expressions on both sides of equation

$\frac{-54}{11}+\frac{8}{11}$ $= -2\left(\frac{23}{11}\right)$

$-\frac{46}{11}$ $= -\frac{46}{11}$

QED

Example: Solve for y: $\frac{y-3}{5}-\frac{y+1}{2}=\frac{2y+1}{10}$

Since the Least Common Denominator (a.k.a. LCM or LCD) is 10, we will use the Multiplication Property of equality with $c=10$ in order to clear fractions. Thus we obtain:

$10\cdot\left[\frac{y-3}{5}-\frac{y+1}{2}\right]=\left[\frac{2y+1}{10}\right]\cdot 10$

$2(y-3)-5(y+1)=2y+1$

$2y-6-5y-5 \quad =2y+1$ Distributive Property

$-3y-11 \quad = 2y+1$ Combine Like Terms

-5y =12 Addition Property of Equality (used twice to isolate y!)

y $= -\frac{12}{5}$ Multiplicative Property of Equality

We will not check every problem but the student is encouraged to do so when applicable.

QED

Remark: "*Clearing fractions*" is a common technique which refers to immediately multiplying both sides of a rational equation by the LCD of all denominators in all terms on both sides of the equation, yielding an equation *with no fractions after the first step*! This efficient technique is far more efficient than working with the fractions through the entire problems because it will highly reduce the chance of arithmetic errors.

We can handle a linear equation involving decimals similarly by eliminating the decimals *on the first step, yielding an equation with no decimals*. We use the multiplication property of equality by multiplying both sides of the equation by the power of ten determined by the *maximum* number of places needed to eliminate decimals in each term on both sides of the equation. The following equation will help illustrate this idea*:*

Example: Solve for p: $.25p+.1=3$

$.25p+.1=3$ (Note: The maximum number of places needed to eliminate the decimals is 2)

$25p+10=300$ Multiplicative property of Equality with c = $10^2=100$

25p = 290 Addition Property of Equality with c = -10

p = $\frac{290}{25}$ Addition Property of Equality with c = -10 p $= \dfrac{58}{5}$ Reduce by dividing the numerator and denominator by $gcf(290,25)=5$

Note: In the last step we multiplied the numerator and denominator by $\frac{1}{5}$. Hence, we can view reducing a fraction (or rational expression) to lowest terms as essentially multiplying the fraction by 1 (but in a specially strategically chosen form)!

QED

Solving Literal Equations For a Specified Variable

*** Technological Resources:** Khanacademy.org

Definition:

A *literal equation* is an equation that involves several variables.

There are many literal equations that relate various quantities to one and other.

Example:

The following are examples of literal equations:

a. $C=\frac{5}{9}(F-32)$, where C is the temperature in degrees Celsius and F in degrees Fahrenheit.

b. $A=\frac{1}{2}h(b_1+b_2)$, where A is the area of a parallelogram with height h and bases b_1 and b_2

QED

Example:

a. Solve for F: $C=\frac{5}{9}(F-32)$.

b. Find the temperature in degrees Fahrenheit if it is 25° Celsius?

Solution:

a.

$$C = \frac{5}{9}(F - 32)$$

$$\frac{9}{5}C = F - 32 \qquad \text{Multiply both sides by } \frac{9}{5}$$

$$F - 32 = \frac{9}{5}C \qquad \text{Symmetric property of Equality}$$

$$F = \frac{9}{5}C + 32 \qquad \text{Additive Property of Equality}$$

b. $F(25) = \frac{9}{5}(25) + 32 = 45 + 32 = 77^{\circ} F$

QED

Solving Linear Inequalities

* **Technological Resources:** Linear41.ggb,linear42.ggb, linear43.ggb, linear44.ggb

** **Technological Resources:** Khanacademy.org

Theorem (Additive Property of Inequalities): If $a < b$, and c $\in \mathbb{R}$, then $a + c < b + c$

Proof: Since $a < b$, a lies to the left of b on the real number line. When we add c to both sides of the inequality, it is clear that $a + c$ lies to the left of $b + c$ on the real number line.

(Note: This property clearly generalizes to include the case of strict inequality)

Theorem (Multiplicative Property of Inequalities): If $a < b$, c $\in \mathbb{R}$, and c<0, then $ac > bc$

Proof: Since $a < b$, a lies to the left of b on the real number line. When we multiply both sides of the inequality by $c < 0$, it is clear that ac lies to the right of bc on the real number line.

(Note: This property also generalizes to include the case of strict inequality)

This property states that if you multiply or divide both sides of an inequality by a negative real number, then the direction of the inequality symbol changes. This property is most useful for linear inequalities because to solve linear inequalities we isolate the variable as we did with linear equations, while assuring to obey multiplicative property of inequalities at the same time.

With higher order inequalities (i.e. quadratic, cubic, etc.) we generally test intervals, making this property less valuable.

Example:

Solve: $-5x+3<18$

$-5x+3<18$	
$-5x<15$	Additive Property of Inequalities
$x>-3$	Multiplicative Property of Inequalities
[-3,∞)	Express answer in interval notation

QED

Compound Linear Inequalities

*** Technological Resources:** Khanacademy.org

Example:

Solve: $3x+2\leq-11$ and $3x+2>-16$

$3x+2\leq-11$	and	$3x+2>-16$	Solve two inequalities seperately
$3x\leq-13$	and	$3x>-18$	
$x\leq-\frac{13}{3}$	and	$x>-6$	
$x>-6$	and	$x\leq-\frac{13}{3}$	
$(-6, -\frac{13}{3}]$			Apply Intersection and write final answer in interval notation

QED

Note: With "And," we take the intersection of the two solution sets, one from each inequality.

In the above example, it is important to point out to the student that a compound inequality that uses "And" can also be written as one compound inequality $-16 < 3x + 2 \le -11$. We can also solve it from this form as shown in the following example.

Example:

Solve: $-16 < 3x + 2 \le -11$

(Note: This is the same example as the previous example but using different notation!)

$-16 < 3x + 2 \le -11$

$-18 < 3x \le -13$ Operate on "All three sides" of the compound inequality

$-6 < x \le -\dfrac{13}{3}$

$(-6, -\dfrac{13}{3}]$ Express final answer in interval notation.

QED

Example:

Solve: $-3x + 1 \le -5$ or $-3x + 1 > 3$

$-3x + 1 \le -5$ or $-3x + 1 > 3$ Solve each inequality seperately

$-3x \le -6$ or $-3x > 2$

$x \ge 2$ or $x < -\dfrac{2}{3}$ Inequality symbol "Flips" by the Mult. Property of Inequalities

$x < -\dfrac{2}{3}$ or $x \ge 2$ Union is commutative

$\left(-\infty, -\dfrac{2}{3}\right) \cup [2, \infty)$ Express final answer in interval notation

QED

Note: With "Or," we take the union of the two solution sets of the inequalities.

Absolute Value Equations

Suppose $|3x-1|=5$. This statement would be true if $3x-1=5$ or $3x-1=-5$. Thus, we would create two equations. However, notice since the compound statement is a disjunction (i.e. "Or"), we would solve this problem in the following way:

Example:

Solve for x : $|3x-1|=5$

$\lvert 3x-1\rvert=5$	
$3x-1=5$ or $3x-1=-5$	Create two equations
$3x=6$ or $3x=-4$	Solve each equation seperately
$x=2$ or $x=-\frac{4}{3}$	Take the dysjunction of the two solution sets
$x=\left\{-\frac{4}{3},2\right\}$	

QED

The previous example leads to the following theorem:

Theorem:

Suppose $c\geq 0$. Then $|x|=c$, iff $x=c$ or $x=-c$

Proof:

Suppose $c\geq 0$. If $x\neq c$ and $x\neq -c$, then $|x|\neq|c|$. Also, if $x=c$ or $x=-c$, then clearly $|x|=c$

It is important to point out that this particular theorem states that we can solve an absolute value equation by isolating the absolute value term and then creating the disjunction of the two equations obtained by setting the inside of the absolute value equal to the other side of the equation and the negative of that quantity.

Caution: This theorem can only be applied when $c\geq 0$. You will see in the following example that if this is not the case, there exists no solution.

Example:

Solve : $|-2x+5|+4=3$

$|-2x+5|+4=3$

$|-2x+5| = -1$ Isolate the absolute value term

No Solution , or $\varnothing$ c < 0

QED

Example:

Solve: $4|-3x+1|-3=5$

$4|-3x+1|-3=5$

$4|-3x+1|=8$ Additive Property of Equality

$|-3x+1|=2$ Multiplicative Property of Equality

$-3x+1=2$ or $-3x+1=-2$ Create two equations using disjunction

$-3x=1$ or $-3x=-3$ Additive Property of Equality

$x=-\frac{1}{3}$ or $x=-1$ Multiplicative Property of Equality

QED

Absolute Value Inequalities in One Variable

*** Technological Resources:** abs_value_inequality.htm, abs_value_inquality.ggb, absolute_value.htm, absolute_value.ggb, absval4.ggb

**** Technological Resources:** Khanacademy.org

Definition: Let c be a positive real number. Then,

i.) If $|x|<c$, then $x<c$ and $x>-c$ (or equivalently, -c<x<c)

ii.) If $|x|>c$, then $x>c$ or $x<-c$

Remark: It is clear the same result would follow for strict inequality in either direction (i.e. $\leq$ or $\geq$)

Notice that there is no equivalent way of writing the disjunction as there is for conjunction, we must isolate the absolute value on one side of the inequality before applying the above definition to solve a problem. Also, the restriction that c is positive is very important as we will see later that unusual solution sets follow if $c \le 0$

Also, it is important to note that property (i) of the definition yields a final answer that is generally between two real values (i.e. the intersection of two linear inequalities) and property (ii) yields a final answer that generally contains a split interval (i.e. the union of two linear inequalities).

In general, x can be a function of x or function of any variable and c can be any positive real number. So, the following examples will help illustrate the previous definition:

Example: Solve for x: $2|2x-5|-4 \le 6$.

$2\|2x-5\|-4 \le 6$			Isolate the absolute value
$2\|2x-5\| \le 10$			Additive property of inequalities with c=4
$\|2x-5\| \le 5$			Multiplicative property of inequalities with c=$\frac{1}{2}$
$2x-5 \le 5$	and	$2x-5 \ge -5$	Case (i) of above definition with c=5
$2x \le 10$	and	$2x \ge 0$	Additive property of inequalities
$x \le 5$	and	$x \ge 0$	Multiplicative property of inequalities
$x \ge 0$	and	$x \le 5$	Commutative property of set intersection (i.e. conjunction)
$x = [0,5]$			Express solution set in interval notation

QED

At the crucial points $x=0$ and $x=5$, we should obtain equality in the above problem. However, we can better check our solution by choosing "test points" between each crucial point and observing that the given inequality will be true for points in our solution set (i.e. $[0,5]$) and false otherwise!

Example: Solve for x: $-3|2x-5|-10 < -16$.

$-3\|2x-5\|-10<-16$	Isolate the absolute value
$-3\|2x-5\|<-6$	Additive property of inequalites
$\|2x-5\|>2$	Mult. property of Inequalities (since $c=-3<0,$ the inequlity symbol flips!)
$2x-5>2$ or $2x-5<-2$	Case (ii) of above definition with $c=2$
$2x>7$ or $2x<3$	Additive property of inequality
$x>\frac{7}{2}$ or $x<\frac{3}{2}$	Multiplicative property of inequalties
$x<\frac{3}{2}$ or $x>\frac{7}{2}$	Commutative property of set union (i.e. disjuction)
$\left(-\infty,\frac{3}{2}\right)\cup\left(\frac{7}{2},\infty\right)$	Express Answer in interval notation

QED

Special Cases ($c \le 0$ in the above definition): What kind of solutions arise?

Technological Resource: The instructor should illustrate this special case using abs_value_inequlaity.htm or abs_value_inequality.ggb.

How do we handle the special cases that arise when solving inequalities that take on the form given in the above definition on absolute value inequalities but with $c \le 0$. Recall, in the given definition above, we required c to be a positive real number.

Example: Solve for x: $|3x+2|<-2$

Given: $|3x+2|<-2$

Since the left hand side of the inequality is always non-negative and the right hand side of the inequality is negative. There is no solution. That is, how can you have a non-negative number (i.e. positive or zero) less than -2. So, there is no real number x that will satisfy the above inequality and we conclude that there is no solution, or the null set , $\varnothing$ is the answer.

QED

Example: Solve for x: $|3x+2| \le 0$

Since the left hand side of the inequality is always positive or zero and the right hand side is zero, the only time the inequality will be satisfied is when the left hand side is identically zero. So, we obtain:

$3x+2=0$ or

$x=-\frac{2}{3}$

QED

Example: Solve for x: $\frac{1}{2}|3x+2|+5>2$

$\frac{1}{2}\|3x+2\|+5>2$	Clear Fractions
$\|3x+2\|+10>4$	Multiplicative property of Inequalities with $c=2$
$\|3x+2\|>-6$	Additive property of inequalities with c = -10
$(-\infty,\infty)$ or $\{x\|x\in\mathbb{R}\}$	Since the left side of the inequality is positive, and thus, will always be greater than a negative number!

QED

Applications

*** Technological Resources:** Khanacademy.org

Example:

A car rental company charges $55 per day for car rental. They also charge an additional $.15 for each mile driven.

a. Find a linear function C(x) that represents the cost to rent the car for one day and drive x miles.

b. How much would it cost to rent the car for one day and drive 200 miles?

c. How many miles can you drive the car during a one day rental if you have exactly $70 to spend on car rental?

Soltuion:

a. $c(x) = 55 + .15x$

b. $c(200) = 55 + .15(200) = 55 + 30 = \$85$

c. We must find x when $c(x) = 70$. So,

$c(x) = 55 + .15x$	
$70 = 55 + .15x$	Sustitute
$.15x = 15$	Addition property of equality
x=100 miles	Multiplication property of equality

QED

Example:

A company's makes a profit when its revenue exceeds its costs. Suppose $C(x) = 10x + 50$ is the cost for the company to produce x units. Also, suppose $R(x) = 12x$ is the revenue that the company makes for selling x units. At least how many units must be produced and sold in order to make a profit?

Solution:

In order for the company to make a profit, the revenue must exceed the cost. In mathematical terminology that means, $12x > 10x + 50$. So solving this inequality yields:

$12x > 10x + 50$	
$2x > 50$	Additive Property of Inequaltiy
x>25	Multiplicative Property of Inequality

Notice that in the original word problem, x is a Natural number and the next natural number strictly greater than 25 is 26. Therefore, the company must produce and sell at least 26 units in order to make a profit.

Note: When x = 25, the cost function equals the revenue function and this is the break-even point!

QED