MAS 4300: Abstract Algebra Broward College

Problem Set 7 - Solutions

1. Prove that $z_8 \oplus z_2$ is not isomorphic to $z_4 \oplus z_4$.

Proof:

$z_8 \oplus z_2$ Contains elements of order 8, while $z_4 \oplus z_4$ does not.

QED

2. Prove that the group of complex numbers under addition is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

Proof:

Define a mapping ϕ from $\mathbb{C}$ to $\mathbb{R} \oplus \mathbb{R}$ by $\phi(a+bi) = (a,b)$.

One-to-one:

Suppose $\phi(a+bi) = \phi(a'+b'i)$. Then $(a,b) = (a',b')$. Thus, $a = a'$ and $\mathrm{b} = b'$.

Onto:

For all $(a,b) \in \mathbb{R} \oplus \mathbb{R}$, $a+bi \in \mathbb{C}$ maps to it.

Operation Preserving (O.P.):

$$\phi\big((a+bi)+(a'+b'i)\big) = \phi\big((a+a')+(b+b')i\big)$$
$$= (a+a', b+b') = (a,b)+(a',b') = \phi(a+bi)+\phi(a'+b'i)$$

QED

3. In $z_{40} \oplus z_{30}$, find two subgroups of order 12.

Solution:

$\langle 10 \rangle \oplus \langle 10 \rangle$ and $\langle 20 \rangle \oplus \langle 5 \rangle$. **QED**

4. Find a subgroup of $z_{12} \oplus z_4 \oplus z_{15}$ that has order 9.

Solution:

$\langle 4 \rangle \oplus \{0\} \oplus \langle 5 \rangle$.

QED

5. Find an isomorphism from z_{12} to $z_4 \oplus z_3$.

Solution:

Define $\begin{array}{l} \phi : \mathbb{Z}_{12} \to (\mathbb{Z}_4 \oplus \mathbb{Z}_3) \text{ by} \\ \phi : x \to (x \bmod 4, x \bmod 3) \end{array}$

One-to-one:

Suppose $\phi(x_1) = \phi(x_2)$. Then, $(x_1 \bmod 4, x_1 \bmod 3) = (x_2 \bmod 4, x_2 \bmod 3)$. Thus,

$x_1 \bmod 4 = x_2 \bmod 4$ for $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{Z}_4$ and $x_1 \bmod 3 = x_2 \bmod 3$ for $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{Z}_3$. So, $x_1 = x_2$.

Onto:

Take $(s,t) \in (\mathbb{Z}_4 \oplus \mathbb{Z}_3)$. Let $s = x \bmod 4$ and $t = x \bmod 3$. Now, the unique $x \in \mathbb{Z}_{12}$ satisfying both of the previous equations simultaneously maps to $(s,t) \in (\mathbb{Z}_4 \oplus \mathbb{Z}_3)$.

Operation Preserving (O.P.):

$$\begin{aligned} \phi(x+y) &= ((x+y) \bmod 4, (x+y) \bmod 3) \\ &= (x \bmod 4, x \bmod 3) + (y \bmod 4, y \bmod 3) = \phi(x) + \phi(y) \end{aligned}$$

QED

6. Suppose that ϕ is an isomorphism from $z_{.3} \oplus z_5$ to z_{15} and $\phi(2,3) = 2$. Find an element in

$z_3 \oplus z_5$ that maps to 1.

Solution:

Since $\phi(2,3) = 2$, we have $8\phi(2,3) = 16 = 1 \pmod{15}$. Now, since ϕ is operation preserving, we have:

$$1 = 8\phi(2,3) = \phi(2,3) + \phi(2,3) + ... + \phi(2,3)$$
$$= \phi((2,3) + (2,3) + ... + (2,3)) = \phi(16,24) = \phi(1,4)$$

(Note: $1 = 1 \bmod 3$ and $4 = 4 \bmod 5$).

So, $\phi(1,4) = 1$.

Alternate Solution:

$\phi : (Z_3 \oplus Z_5) \to Z_{15}$. Now, $(2,3) = (2,-2)$ maps to 2. Also, $(1,-1) = (1,4)$ and $(1,-1)$ a generator of $(Z_3 \oplus Z_5)$. So, $\phi(1,4) = 1$.

QED

7. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} | \, a,b,d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2,\mathbb{R})$?

Solution:

No. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then, A is in H and B is in $GL(2,\mathbb{R})$,

but $BAB^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \notin H$. So H is not Normal in $GL(2,\mathbb{R})$ by the normal subgroup test.

QED

8. Prove that a factor group of a cyclic group is cylic.

Proof:

If G is generated by a, then G/H is generated by aH. Symbolically, that is if $G=\langle a\rangle$, then $G/H=\langle aH\rangle$.

QED

9. Prove that a factor group of an abelian group is abelian.

Proof:

$(aH)(bH)=(ab)H=(ba)H=(bH)(aH)$.

(Note: The first and third equalities are a result of the definition of the product in factor groups and the middle equality is due to the underlying group being abelian.)

QED

10. What is the order of the element $14+\langle 8\rangle$ in the factor group $Z_{24}/\langle 8\rangle$?

Solution:

$\mathbb{Z}_{24}/\langle 8\rangle=\{a+\langle 8\rangle \mid a\in\mathbb{Z}_{24}\}=\{0+\langle 8\rangle,1+\langle 8\rangle,...7+\langle 8\rangle\}$.

Computing powers of $14+\langle 8\rangle$, we obtain:

$(14+\langle 8\rangle)^2=(14+\langle 8\rangle)(14+\langle 8\rangle)=(14+14)+\langle 8\rangle=28+\langle 8\rangle=4+\langle 8\rangle\neq e$.

$(14+\langle 8\rangle)^3=(4+\langle 8\rangle)(14+\langle 8\rangle)=18+\langle 8\rangle\neq e$.

$(14+\langle 8\rangle)^4=(18+\langle 8\rangle)(14+\langle 8\rangle)=32+\langle 8\rangle=(8+\langle 8\rangle)=0+\langle 8\rangle=e$.

So, $|14+\langle 8\rangle|=4$.

QED