MAS 4300: Abstract Algebra Broward College

Problem Set 6 - Solutions

Directions: Work all of the following problems.

1. Prove or Disprove that $U(20)$ and $U(24)$ are isomorphic.

   Solution: They are not isomorphic since U(20) has three elements of order 2, whereas U(24) has seven elements of order 2.

   QED

2. Let $G = \{a + b\sqrt{2} \mid a, b \text{ rational}\}$ and $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a, b \text{ rational} \right\}$. Show that G and H are isomorphic under addition. Does your isomorphism preserve multiplication as well?

   Solution:

   Let $\phi : \left(a + b\sqrt{2}\right) \to \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$. This function is clearly a bijective map. To show O.P., suppose $x = a + b\sqrt{2}$ and $y = a' + b'\sqrt{2}$. Then

   $$\phi(x+y) = \phi\left(\left(a + b\sqrt{2}\right) + \left(a' + b'\sqrt{2}\right)\right) = \phi\left((a + a') + \left((b + b')\sqrt{2}\right)\right)$$
   $$= \begin{bmatrix} a + a' & 2b + 2b' \\ b + b' & a + a' \end{bmatrix} = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} a' & 2b' \\ b' & a' \end{bmatrix} = \phi(x) + \phi(y).$$

   Yes, the given isomorphism also preserves multiplication!

   QED

3. Let $R^n = \{(a_1, a_2, ..., a_{n-1}, a_n) \mid a_i \in R\}$. Show that the mapping $\phi : (a_1, a_2, ..., a_n) \to (-a_1, -a_2, ..., -a_n)$ is an automorphism of the group R^n under componentwise addition. Describe the action of the *inversion* , ϕ , geometrically.

Solution:

$(-a_1,...,-a_n)=(-b_1,...,-b_n)$ implies $(a_1,...,a_n)=(b_1,...,b_n)$., so ϕ is 1-1. Now, for any $(a_1,...,a_n)$, we have $\phi(-a_1,...,-a_n)=(a_1,...,a_n)$. So, ϕ is onto. To show ϕ O.P., we have

$$\begin{aligned}&\phi((a_1,...,a_n)+(b_1,...,b_n))=\phi(a_1+b_1,...,a_n+b_n)=(-(a_1+b_1),...-(a_n+b_n))\\&=(-a_1,...,-a_n)+(-b_1,...,-b_n)=\phi(a_1,...,a_n)+\phi(b_1,...,b_n)\end{aligned}.$$

QED

4. Let $G=\{0,\pm2,\pm4,\pm6,...\}$ and $H=\{0,\pm3,\pm6,\pm9,...\}$. Show that G and H are isomorphic groups under addition. Generalize the case when $G=\langle m\rangle$ and $H=\langle n\rangle$.

Proof:

The mapping $\phi(x)=\frac{3}{2}x$ is an isomorphism from G onto H (easy to verify). When $G=\langle m\rangle$ and $H=\langle n\rangle$, the mapping $\phi(x)=\frac{3}{2}x$ is an isomorphism from G to H.

QED

5. Show that every automorphism ϕ of the rational numbers Q under addition to itself has the form $\phi(x)=x\phi(1)$.

Proof:

$$\phi(1)=\phi\left(\frac{1}{n}+\frac{1}{n}+...\frac{1}{n}\right) \qquad \text{n of these}$$

$$=n\phi\left(\frac{1}{n}\right)$$

(Cont.)

Proof (Cont.)

So, $\phi\left(\frac{1}{n}\right)=\frac{1}{n}\phi(1)$. Now , for the same reasoning $(i.e.$ since ϕ additive$)$

$\phi\left(\frac{m}{n}\right)=m\phi\left(\frac{1}{n}\right)$. So, $\phi\left(\frac{m}{n}\right)=m\phi\left(\frac{1}{n}\right)=m\left(\frac{1}{n}\phi(1)\right)=\frac{m}{n}\phi(1)$.

QED

6. Let n be a positive integer. Let $H=\{0,\pm n,\pm 2n,\pm 3n,...\}$. Find all left cosets of H in Z. How many are there?

Solution:

There are n cosets. They are $0+\langle n\rangle, 1+\langle n\rangle, 2+\langle n\rangle,....(n-1)+\langle n\rangle$.

QED

7. Suppose that a has order 15. Find all of the left cosets of $\langle a^5\rangle$ in $\langle a\rangle$.

Solution:

$|\langle a^5\rangle|=3$ so there are $\frac{15}{3}=5$ cosets. They are $\langle a^5\rangle, a\langle a^5\rangle, a^2\langle a^5\rangle, a^3\langle a^5\rangle, a^4\langle a^5\rangle$.

QED

8. Let C^* be the group of nonzero complex numbers under multiplication and let $H=\{a+bi\in C^* \mid a^2+b^2=1\}$. Give a geometric description of the coset $(c+di)H$.

Solution:

The coset containing $c+di$ is the circle with center at the origin and radius $\sqrt{c^2+d^2}$

QED

9. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K|=42$ and $|G|=420$, what are the possible orders of H?

Solution:

84 or 210.

QED

10. Prove that the order of $U(n)$ is even when $n > 2$.

Proof:

($n-1$) in U(n) has order 2. Also, the order of any element divides the order of the group. So, we have that 2 divides the order of the group. Thus, the order of U(n) is even.

QED

11. Suppose $|G| = 8$. Show that G must have an element of order 2.

Proof:

Let $e \neq g \in G$. If $|g| = 8$, then $|g^4| = 2$. If $|g| = 4$, then $|g^2| = 2$.

QED