MAS 4300: Abstract Algebra Broward College

Problem Set 5 - Solutions

Directions: Work all of the following problems.

1. How many elements of order 5 are in s_7? You must justify your answer.

   Solution:

   An element of order 5 in s_7 takes the form $(a_1a_2a_3a_4a_5)$. Now there are $P(7,5)=21$ ways to do this. However, any rotation of such an element is the same $\left[i.e.\ (a_2a_3a_4a_5a_1) \text{ is one such rotation}\right]$. Since there are 5 such rotations for each, we can see that there are $\frac{P(7,5)}{5}=\frac{7\cdot6\cdot5\cdot4\cdot3}{5}=504$ such ways.

   QED

2. Prove that (1234) is not the product of 3-cycles.

   Proof:

   Since is odd and any product of three-cycles is even, we see that this cannot happen. (Recall: a permutation's parity is independent of its representation).

   QED

3. Let $\beta=(123)(145)$. Write β^{99} in disjoint cycle form.

   Solution:

   First, the order of β is 5, so $\beta^5=e$. Also, $(123)(145)=(14523)$.

   So, $\beta^{99}=\left(\beta^5\right)^{20}B^{-1}=e^{20}B^{-1}=eB^{-1}=\beta^{-1}=(32541)$. **QED**

4. Let $\beta=(1,3,5,7,9,8,6)(2,4,10)$. What is the smallest positive integer n for which $\beta^n=\beta^{-5}$? You must justify your work.

   Solution:

Since $|\beta| = \text{LCM}(7,3) = 21$, we know that $B^{21} = e$. So,

$B^{-5} = B^{21} \cdot B^{-26} = e \cdot B^{-26} = B^{-26} = B^{5} = B^{16}$. So, $n = 16$.

QED

5.

a. Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of s_5.

b. How many elements are in H? Is your argument valid in S_n for any n? How many elements are in H in this case?

Solution:

(a)

Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$

Let $\beta, \gamma \in H$. Then, $\beta\gamma(1) = \beta(\gamma(1)) = \beta(1) = 1$ and $\beta\gamma(3) = \beta(\gamma(3)) = \beta(3) = 3$. Thus, H closed. So, by the finite subgroup test, H is a subgroup of s_5.

(b)

Since β fixers 1 and 3, we see that we have to permute three other elements. This can be done in $3! = 6$ ways. So, $|H| = 6$. This proof is valid for all $n \geq 3$ and in general, $|H| = (n-2)!$ since the image of exactly two elements is pre-determined.

QED

6. Find an isomorphism from the group of integers under addition to the group of even integers under addition.

Solution:

Let $\phi(n) = 2n$. Now, ϕ is onto since for all n in the set of even integers, we have $\frac{n}{2} \in Z$ maps to it. ϕ is one-to-one since $\phi(m) = \phi(n)$ implies $2m = 2n$, or $m = n$. Lastly, ϕ is operation preserving since $\phi(m+n) = 2(m+n) = 2m + 2n = \phi(m) + \phi(n)$. Thus, ϕ an automorphism.

QED

7. Let R^+ be the group of positive real numbers under multiplication. Show that the mapping $\phi(x) = \sqrt{x}$ is an automorphism of R^+.

Proof:

Let $\phi(x) = \sqrt{x}$ be a mapping from R^+ to R^+. Now, ϕ is onto since $\phi^{-1}(n) = n^2$ for all n in the positive Real numbers. That is, $\phi(n^2) = \sqrt{n^2} = |n| = n$. ϕ is one-to-one since $\phi(a) = \phi(b)$ implies $\sqrt{a} = \sqrt{b}$, or simply $a = b$ (by squaring both sides). Finally, ϕ is operation preserving since $\phi(ab) = \sqrt{ab} = \sqrt{a}\sqrt{b} = \phi(a)\phi(b)\ \forall\ a, b \in R^+$. Thus, ϕ an automorphism from R^+ to R^+.

QED

8. Show that $U(8)$ is not isomorphic to $U(10)$.

Proof:

$U(10)$ is cyclic, but $U(8)$ is not.

QED

9. Show that $U(8)$ is isomorphic to $U(12)$.

Proof:

Define ϕ from $U(8)$ to $U(12)$ by $\phi(1)=1, \phi(3)=5, \phi(5)=7, \phi(7)=11$. This map is clearly a bijection. To see that ϕ is operation-preserving, observe that:

$$\phi(1\cdot a)=\phi(a)=\phi(a)\cdot 1=\phi(a)\phi(1)=\phi(1)\phi(a)$$
$$\phi(3\cdot 5)=\phi(7)=11=5\cdot 7=\phi(3)\phi(5)$$
$$\phi(3\cdot 7)=\phi(5)=7=5\cdot 11=\phi(3)\phi(7)$$
$$\phi(3\cdot 7)=\phi(5)=7=5\cdot 11=\phi(3)\phi(7) \quad \text{and}$$
$$\phi(5\cdot 7)=\phi(3)=5=7\cdot 11=\phi(5)\phi(7)$$

So, the above with the fact that $U(n)$ is abelian shows that
$\phi(ab)=\phi(a)\phi(b)\ \forall\ a,b\in U(8)$.

QED

10. Show that the mapping $a \to \log_{10} a$ is an isomorphism from R^+ under multiplication to R under addition.

Proof:

Suppose $\phi : (R^+,\cdot)\to(R,+)$ by $\phi(x)=\log_{10}(x)$. Clearly ϕ is a bijection due to

properties of real numbers. Now, ϕ is operation preserving since

$\phi(ab)=\log_{10}(ab)=\log_{10}(a)+\log_{10}(b)=\phi(a)+\phi(b)$.

QED

11. Let G be a group. Prove that the mapping $\alpha(g)=g^{-1}$ for all $g\in G$ is an automorphism if and only if G is abelian.

Proof:

Suppose $\alpha(g) = g^{-1}$ is an automorphism on G. Then $\alpha(gh) = (gh)^{-1}$.

$(\rightarrow)$

Suppose G an automorphism. Then, $\alpha(gh) = \alpha(g)\alpha(h) = g^{-1}h^{-1} \ \forall\ g,h \in G$. So,

$h^{-1}g^{-1} = (gh)^{-1} = \alpha(gh) = g^{-1}h^{-1}$ (the last equality follows from above).

So, we have $h^{-1}g^{-1} = g^{-1}h^{-1}$. Taking the inverse of both sides, we obtain

$(gh)^{-1} = (hg)^{-1}$ and we have established that $gh = hg \ \forall\ g,h \in G$ (since inverses are unique).

$(\leftarrow)$

Suppose G is abelian. Then for all $g,h \in G$, we have
$\alpha(gh) = (gh)^{-1} = h^{-1}g^{-1} = \alpha(h)\alpha(g) = \alpha(g)\alpha(h)$.

Thus, α an automorphism iff G abelian.

QED

12. Suppose that $\phi : Z_{50} \rightarrow Z_{50}$ is an automorphism with $\phi(11) = 13$. Find a formula for $\phi(x)$.

Proof:

Since $13 = \phi(11) = \phi(1+1+...+1) = \phi(1)+\phi(1)...+\phi(1) = 11\phi(1)$, we have that .

$\phi(1) = 11^{-1} \cdot 13 = 41 \cdot 13 = 33$ (Note: We can find $11^{-1} (\bmod 50)$ by using the Euclidean Algorithm). So, $\phi(x) = \phi(1 \cdot x) = \phi(1) \cdot \phi(x) = 33x$. So, $\phi(x) = 33x$.

[Note: We can check to see that $\phi(11) = 33 \cdot 11 = 363 = 13$]

QED