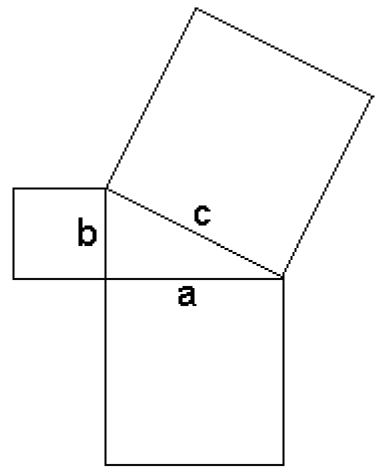
Pythagorean Theorem - The Many Proofs



Professor R. Smullyan in his book *5000 B.C. and Other Philosophical Fantasies* tells of an experiment he ran in one of his geometry classes. He drew a right triangle on the board with squares on the hypotenuse and legs and observed the fact the the square on the hypotenuse had a larger area than either of the other two squares. Then he asked, "Suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?" Interestingly enough, about half the class opted for the one large square and half for the two small squares. Both groups were equally amazed when told that it would make no difference.

The *Pythagorean* (or *Pythagoras'*) *Theorem* is the statement that the sum of (the areas of) the two small squares equals (the area of) the big one.

In algebraic terms, $\mathbf{a^2 + b^2 = c^2}$ where **c** is the hypotenuse while **a** and **b** are the legs of the triangle.

The theorem is of fundamental importance in Euclidean Geometry where it serves as a basis for the definition of distance between two points. It's so basic and well known that, I believe, anyone who took geometry classes in high school couldn't fail to remember it long after other math notions got thoroughly forgotten.

Below is a collection of 93 approaches to proving the theorem. Many of the proofs are accompanied by interactive Java illustrations.

Remark

1. The statement of the Theorem was discovered on a Babylonian tablet circa 1900-1600 B.C. Whether Pythagoras (c.560-c.480 B.C.) or someone else from his School was the first to discover its proof can't be claimed with any degree of credibility. Euclid's (c 300 B.C.) *Elements* furnish the first and, later, the standard reference in Geometry. In fact Euclid supplied two very different proofs: the Proposition I.47 (First Book, Proposition 47) and VI.31. The Theorem is *reversible* which means that its *converse* is also true. The converse states that a triangle whose sides satisfy $a^2 + b^2 = c^2$ is necessarily right angled. Euclid was the first (I.48) to mention and prove this fact.
2. W. Dunham [*Mathematical Universe*] cites a book *The Pythagorean Proposition* by an early 20th century professor Elisha Scott Loomis. The book is a collection of 367 proofs of the Pythagorean Theorem and has been republished by NCTM in 1968. In the Foreword, the author rightly asserts that the number of algebraic proofs is limitless as is also the number of geometric proofs, but that the proposition admits no trigonometric proof. Curiously, nowhere in the book does Loomis mention Euclid's VI.31 even when offering it and the variants as algebraic proofs 1 and 93 or as geometric proof 230.

In all likelihood, Loomis drew inspiration from a series of short articles in *The American Mathematical Monthly* published by B. F. Yanney and J. A. Calderhead in 1896-1899. Counting possible variations in calculations derived from the same geometric configurations, the potential number of proofs there grew into thousands. For example, the authors counted 45 proofs based on the diagram of proof #6 and virtually as many based on the diagram of #19 below. I'll give an example of their approach in proof #56. (In all, there were 100 "shorthand" proofs.)

I must admit that, concerning the existence of a trigonometric proof, I have been siding with with Elisha Loomis until very recently, i.e., until I was informed of Proof #84.

In trigonometric terms, the Pythagorean theorem asserts that in a triangle ABC, the equality $\sin^2 A + \sin^2 B = 1$ is equivalent to the angle at C being right. A more symmetric assertion is that ΔABC is right iff $\sin^2 A + \sin^2 B + \sin^2 C = 2$. By the sine law, the latter is equivalent to $a^2 + b^2 + c^2 = 2d^2$, where d is the diameter of the circumcircle. Another form of the same property is $\cos^2 A + \cos^2 B + \cos^2 C = 1$ which I like even more.

3. Pythagorean Theorem generalizes to spaces of higher dimensions. Some of the generalizations are far from obvious. Pythagorean theorem serves as the basis of the Euclidean distance formula.
4. Larry Hoehn came up with a plane generalization which is related to the law of cosines but is shorter and looks nicer.
5. The Theorem whose formulation leads to the notion of Euclidean distance and Euclidean and Hilbert spaces, plays an important role in Mathematics as a

whole. There is a small collection of rather elementray facts whose proof may be based on the Pythagorean Theorem. There is a more recent page with a list of properties of the Euclidian diagram for I.47.

6. Wherever all three sides of a right triangle are integers, their lengths form a *Pythagorean triple* (or *Pythagorean numbers*). There is a general formula for obtaining all such numbers.
7. My first math droodle was also related to the Pythagorean theorem. Unlike a proof without words, a droodle may suggest a statement, not just a proof.
8. Several false proofs of the theorem have also been published. I have collected a few in a separate page. It is better to learn from mistakes of others than to commit one's own.
9. It is known that the Pythagorean Theorem is Equivalent to Parallel Postulate.
10. The Pythagorean configuration is known under many names, the Bride's Chair being probably the most popular. Besides the statement of the Pythagorean theorem, Bride's chair has many interesting properties, many quite elementary.
11. The late Professor Edsger W. Dijkstra found an absolutely stunning generalization of the Pythagorean theorem. If, in a triangle, angles α, β, γ lie opposite the sides of length a, b, c, then

    (EWD) $\operatorname{sign}(\alpha + \beta - \gamma) = \operatorname{sign}(a^2 + b^2 - c^2)$,

12. where sign(t) is the *signum* function:

    $\operatorname{sign}(t) = -1$, for $t < 0$,

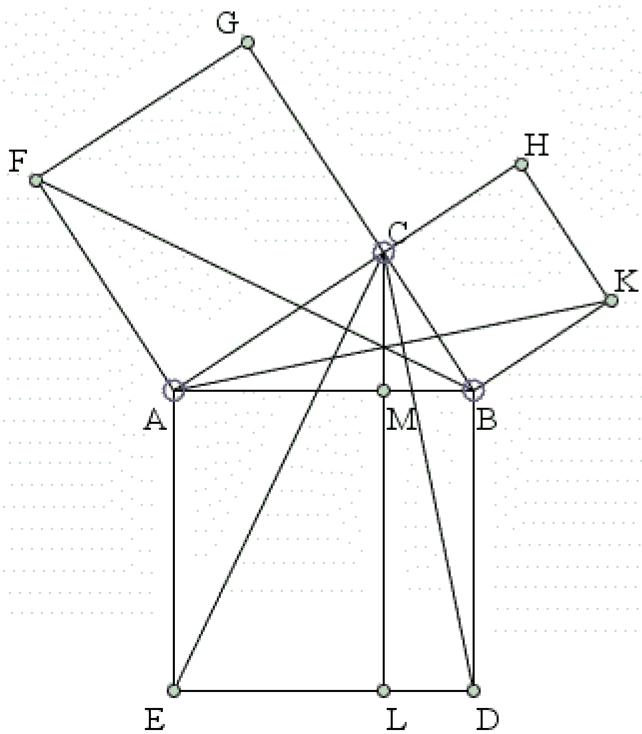
    $\operatorname{sign}(0) = 0$,

    $\operatorname{sign}(t) = 1$, for $t > 0$.

13. The theorem this page is devoted to is treated as "If $\gamma = \pi/2$, then $a^2 + b^2 = c^2$." Dijkstra deservedly finds (EWD) more symmetric and more informative. Absence of transcendental quantities (π) is judged to be an additional advantage. Dijkstra's proof is included as Proof 78 and is covered in more detail on a separate page.
14. The most famous of right-angled triangles, the one with dimensions 3:4:5, has been sighted in Gothic Art and can be obtained by paper folding. Rather inadvertently, it pops up in several Sangaku problems.
15. Perhaps not surprisingly, the Pythagorean theorem is a consequence of various physical laws and is encountered in several mechanical phenomena.

Proof #1

This is probably the most famous of all proofs of the Pythagorean proposition. It's the first of Euclid's two proofs (I.47). The underlying configuration became known under a variety of names, the *Bride's Chair* likely being the most popular.



The proof has been illustrated by an award winning Java applet written by Jim Morey. I include it on a separate page with Jim's kind permission. The proof below is a somewhat shortened version of the original Euclidean proof as it appears in Sir Thomas Heath's translation.

First of all, ΔABF = ΔAEC by SAS. This is because, AE = AB, AF = AC, and

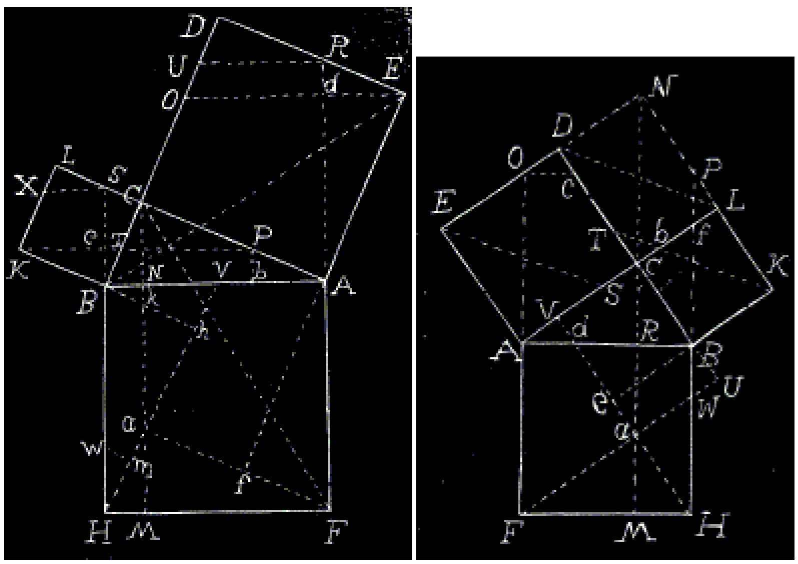
∠BAF = ∠BAC + ∠CAF = ∠CAB + ∠BAE = ∠CAE.

ΔABF has base AF and the altitude from B equal to AC. Its area therefore equals half that of square on the side AC. On the other hand, ΔAEC has AE and the altitude from C equal to AM, where M is the point of intersection of AB with the line CL parallel to AE. Thus the area of ΔAEC equals half that of the rectangle AELM. Which says that the area AC^2 of the square on side AC equals the area of the rectangle AELM.

Similarly, the area BC^2 of the square on side BC equals that of rectangle BMLD.

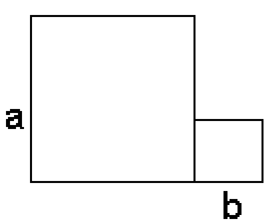
Finally, the two rectangles AELM and BMLD make up the square on the hypotenuse AB.

The configuration at hand admits numerous variations. B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v.4, n 6/7, (1987), 168-170 published several proofs based on the following diagrams

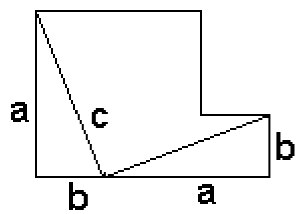


Some properties of this configuration has been proved on the Bride's Chair and others at the special Properties of the Figures in Euclid I.47 page.

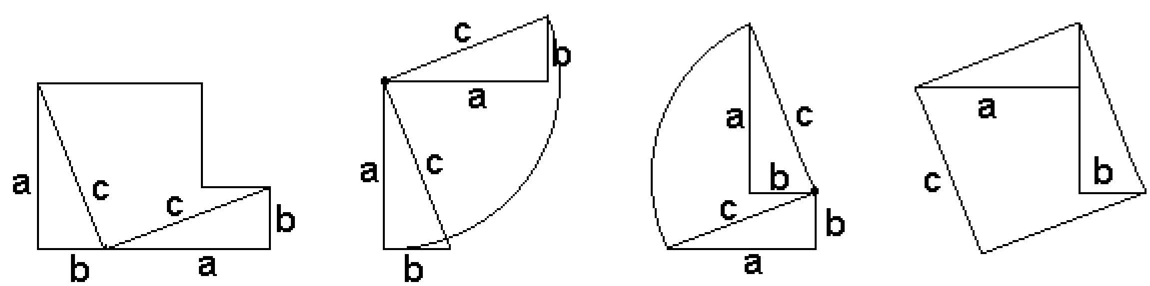
Proof #2



We start with two squares with sides **a** and **b**, respectively, placed side by side. The total area of the two squares is **a²+b²**.



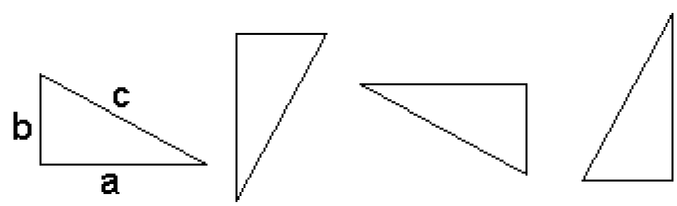
The construction did not start with a triangle but now we draw two of them, both with sides **a** and **b** and hypotenuse **c**. Note that the segment common to the two squares has been removed. At this point we therefore have two triangles and a strange looking shape.



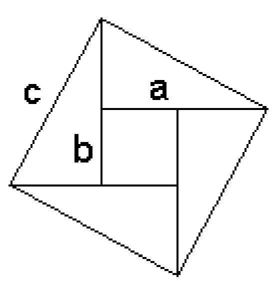
As a last step, we rotate the triangles 90°, each around its top vertex. The right one is rotated clockwise whereas the left triangle is rotated counterclockwise. Obviously the resulting shape is a square with the side c and area **c²**. This proof appears in a [dynamic incarnation].

(A variant of this proof is found in an extant manuscript by Thâbit ibn Qurra located in the library of Aya Sofya Musium in Turkey, registered under the number 4832. [R. Shloming, Thâbit ibn Qurra and the Pythagorean Theorem, *Mathematics Teacher* 63 (Oct., 1970), 519-528]. ibn Qurra's diagram is similar to that in [proof #27]. The proof itself starts with noting the presence of four equal right triangles surrounding a strangely looking shape as in the current proof #2. These four triangles correspond in pairs to the starting and ending positions of the rotated triangles in the current proof. This same configuration could be observed in a proof by tessellation.)

Proof #3



Now we start with four copies of the same triangle. Three of these have been rotated 90°, 180°, and 270°, respectively. Each has area **ab**/2. Let's put them together without additional rotations so that they form a square with side **c**.



The square has a square hole with the side (**a - b**). Summing up its area $(a - b)^2$ and 2**ab**, the area of the four triangles (4·**ab**/2), we get

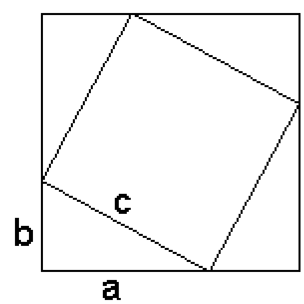
$$c^2 = (a - b)^2 + 2ab$$

$= a^2 - 2ab + b^2 + 2ab$

$= a^2 + b^2$

Proof #4

The fourth approach starts with the same four triangles, except that, this time, they combine to form a square with the side (**a** + **b**) and a hole with the side **c**. We can compute the area of the big square in two ways. Thus

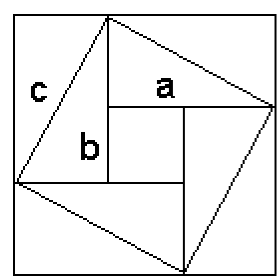


$$(\mathbf{a} + \mathbf{b})^2 = 4 \cdot \mathbf{ab}/2 + \mathbf{c}^2$$

simplifying which we get the needed identity.

A proof which combines this with proof #3 is credited to the 12th century Hindu mathematician Bhaskara (Bhaskara II):

Here we add the two identities



$$\mathbf{c}^2 = (\mathbf{a} - \mathbf{b})^2 + 4 \cdot \mathbf{ab}/2 \text{ and}$$

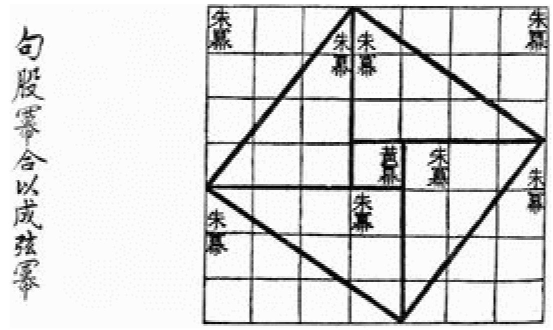
$$\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})^2 - 4 \cdot \mathbf{ab}/2$$

which gives

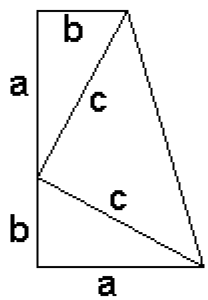
$$2\mathbf{c}^2 = 2\mathbf{a}^2 + 2\mathbf{b}^2.$$

The latter needs only be divided by 2. This is the algebraic proof # 36 in Loomis' collection. Its variant, specifically applied to the 3-4-5 triangle, has featured in the

Chinese classic *Chou Pei Suan Ching* dated somewhere between 300 BC and 200 AD and which Loomis refers to as proof 253.



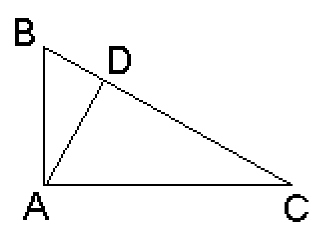
Proof #5



This proof, discovered by President J.A. Garfield in 1876 [Pappas], is a variation on the previous one. But this time we draw no squares at all. The key now is the formula for the area of a trapezoid - *half sum of the bases times the altitude* - (**a** + **b**)/2·(**a** + **b**). Looking at the picture another way, this also can be computed as the sum of areas of the three triangles - **ab**/2 + **ab**/2 + **c**·**c**/2. As before, simplifications yield $a^2 + b^2 = c^2$.

Two copies of the same trapezoid can be combined in two ways by attaching them along the slanted side of the trapezoid. One leads to the proof #4, the other to proof #52.

Proof #6



We start with the original right triangle, now denoted ABC, and need only one additional construct - the altitude AD. The triangles ABC, DBA, and DAC are similar which leads to two ratios:

$AB/BC = BD/AB$ and $AC/BC = DC/AC$.

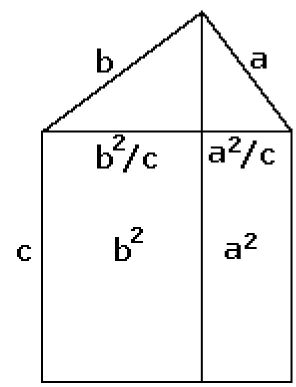
Written another way these become

$AB \cdot AB = BD \cdot BC$ and $AC \cdot AC = DC \cdot BC$

Summing up we get

$$AB \cdot AB + AC \cdot AC = BD \cdot BC + DC \cdot BC$$

$$= (BD+DC) \cdot BC = BC \cdot BC.$$



In a little different form, this proof appeared in the *Mathematics Magazine*, 33 (March, 1950), p. 210, in the Mathematical Quickies section, see *Mathematical Quickies*, by C. W. Trigg.

Taking $AB = a$, $AC = b$, $BC = c$ and denoting $BD = x$, we obtain as above

$a^2 = cx$ and $b^2 = c(c - x)$,

which perhaps more transparently leads to the same identity.

In a private correspondence, Dr. France Dacar, Ljubljana, Slovenia, has suggested that the diagram on the right may serve two purposes. First, it gives an additional graphical representation to the present proof #6. In addition, it highlights the relation of the latter to proof #1.

R. M. Mentock has observed that a little trick makes the proof more succinct. In the common notations, $c = b \cos A + a \cos B$. But, from the original triangle, it's easy to see that $\cos A = b/c$ and $\cos B = a/c$ so $c = b (b/c) + a (a/c)$. This variant immediately brings up a question: are we getting in this manner a trigonometric proof? I do not think so, although a trigonometric function (cosine) makes here a prominent appearance. The ratio of two lengths in a figure is a shape property meaning that it remains fixed in passing between similar figures, i.e., figures of the same shape. That a particular ratio used in the proof happened to play a sufficiently important role in trigonometry and, more generally, in mathematics, so as to deserve a special notation of its own, does not cause the proof to depend on that notation. (However, check Proof 84 where trigonometric identities are used in a significant way.)

Finally, it must be mentioned that the configuration exploited in this proof is just a specific case of the one from the next proof - Euclid's second and less known proof of the Pythagorean proposition. A separate page is devoted to a proof by the similarity argument.

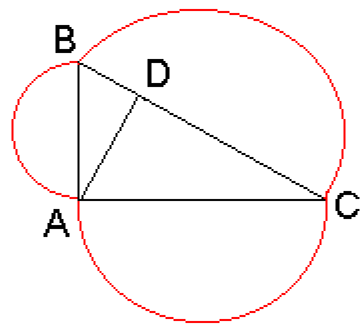
Proof #7

The next proof is taken verbatim from Euclid VI.31 in translation by Sir Thomas L. Heath. The great G. Polya analyzes it in his *Induction and Analogy in Mathematics* (II.5) which is a recommended reading to students and teachers of Mathematics.

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right; I say that the figure on BC is equal to the similar and similarly described figures on BA, AC.

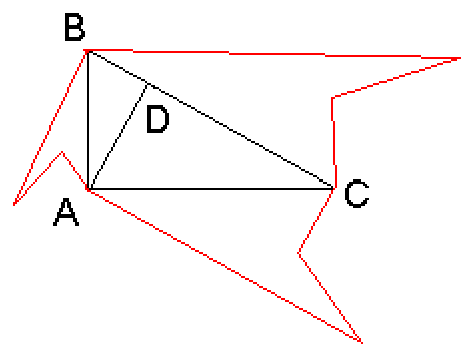
Let AD be drawn perpendicular. Then since, in the right-angled triangle ABC, AD has been drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC adjoining the perpendicular are similar both to the whole ABC and to one another [VI.8].



And, since ABC is similar to ABD, therefore, as CB is to BA so is AB to BD [VI.Def.1].

And, since three straight lines are proportional, as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second [VI.19]. Therefore, as CB is to BD, so is the figure on CB to the similar and similarly described figure on BA.

For the same reason also, as BC is to CD, so is the figure on BC to that on CA; so that, in addition, as BC is to BD, DC, so is the figure on BC to the similar and similarly described figures on BA, AC.



But BC is equal to BD, DC; therefore the figure on BC is also equal to the similar and similarly described figures on BA, AC.

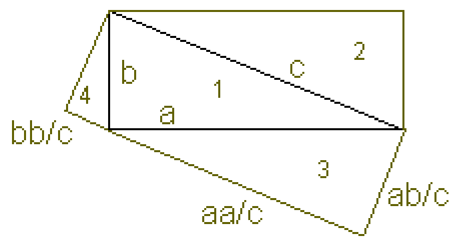
Therefore etc. Q.E.D.

Confession

I got a real appreciation of this proof only after reading the book by Polya I mentioned above. I hope that a [Java applet]() will help you get to the bottom of this remarkable proof. Note that the statement actually proven is much more general than the theorem as it's generally known. (Another discussion looks at VI.31 from a little different angle.)

Proof #8

Playing with the applet that demonstrates the Euclid's proof (#7), I have discovered another one which, although ugly, serves the purpose nonetheless.



Thus starting with the triangle 1 we add three more in the way suggested in proof #7: similar and similarly described triangles 2, 3, and 4. Deriving a couple of ratios as was done in proof #6 we arrive at the side lengths as depicted on the diagram. Now, it's possible to look at the final shape in two ways:

- as a union of the rectangle (1 + 3 + 4) and the triangle 2, or
- as a union of the rectangle (1 + 2) and two triangles 3 and 4.

Equating the areas leads to

$$ab/c \cdot (a^2 + b^2)/c + ab/2 = ab + (ab/c \cdot a^2/c + ab/c \cdot b^2/c)/2$$

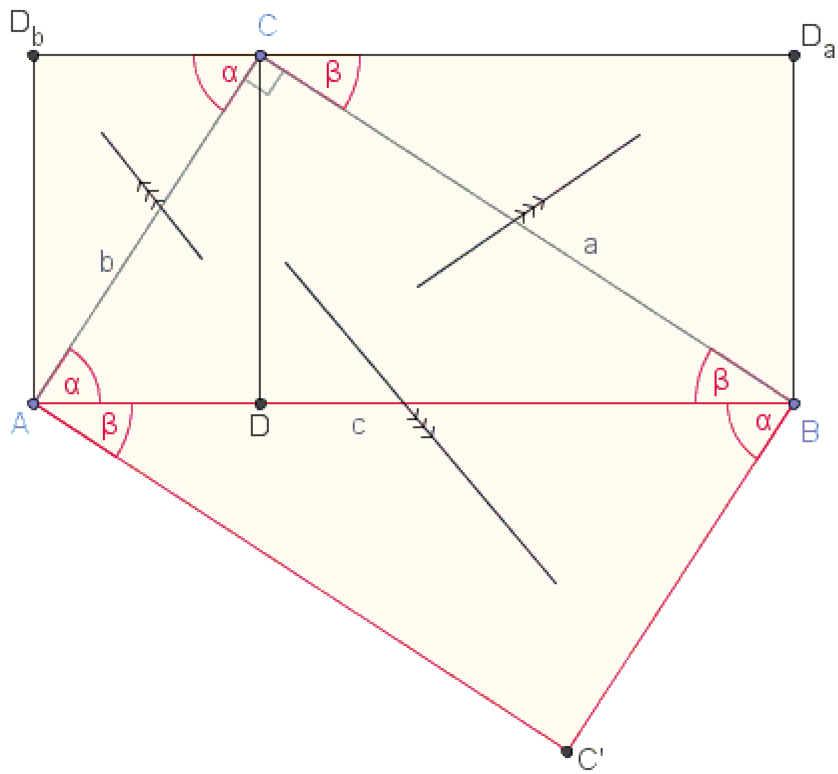
Simplifying we get

$$ab/c \cdot (a^2 + b^2)/c/2 = ab/2, \text{ or } (a^2 + b^2)/c^2 = 1$$

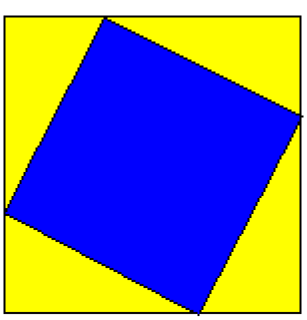
Remark

In hindsight, there is a simpler proof. Look at the rectangle (1 + 3 + 4). Its long side is, on one hand, plain c, while, on the other hand, it's $a^2/c + b^2/c$ and we again have the same identity.

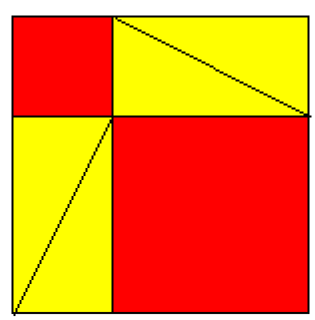
Vladimir Nikolin from Serbia supplied a beautiful illustration:



Proof #9



Another proof stems from a rearrangement of rigid pieces, much like [proof #2](). It makes the algebraic part of [proof #4]() completely redundant. There is nothing



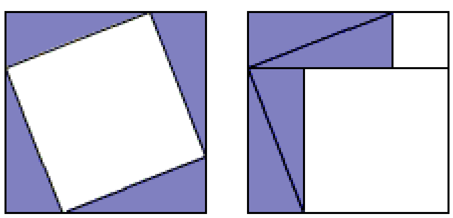
much one can add to the two pictures.

(My sincere thanks go to Monty Phister for the kind permission to use the graphics.)

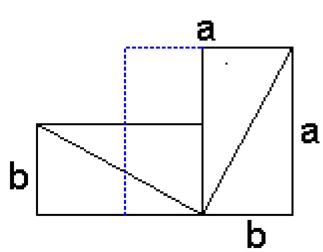
There is an interactive simulation to toy with. And another one that clearly shows its relation to proofs #24 or #69.

Loomis (pp. 49-50) mentions that the proof "was devised by Maurice Laisnez, a high school boy, in the Junior-Senior High School of South Bend, Ind., and sent to me, May 16, 1939, by his class teacher, Wilson Thornton."

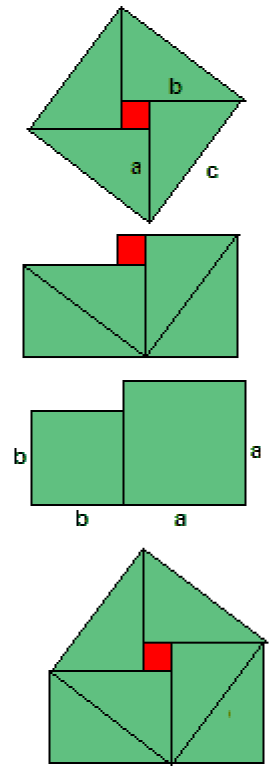
The proof has been published by Rufus Isaac in *Mathematics Magazine*, Vol. 48 (1975), p. 198.



A slightly different rearragement leads to a hinged dissection illustrated by a Java applet.



Proof #10



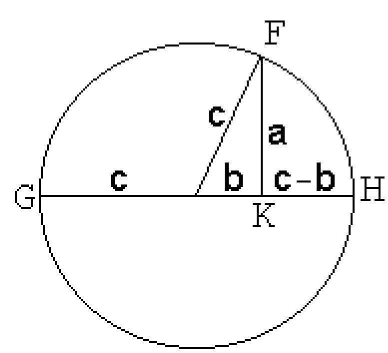
This and the next 3 proofs came from [*PWW*].

The triangles in Proof #3 may be rearranged in yet another way that makes the Pythagorean identity obvious.

(A more elucidating diagram on the right was kindly sent to me by Monty Phister. The proof admits a hinged dissection illustrated by a Java applet.)

The first two pieces may be combined into one. The result appear in a 1830 book *Sanpo Shinsyo - New Mathematics* - by Chiba Tanehide (1775-1849), [H. Fukagawa, A. Rothman, *Sacred Mathematics: Japanese Temple Geometry*, Princeton University Press, 2008, p. 83].

Proof #11

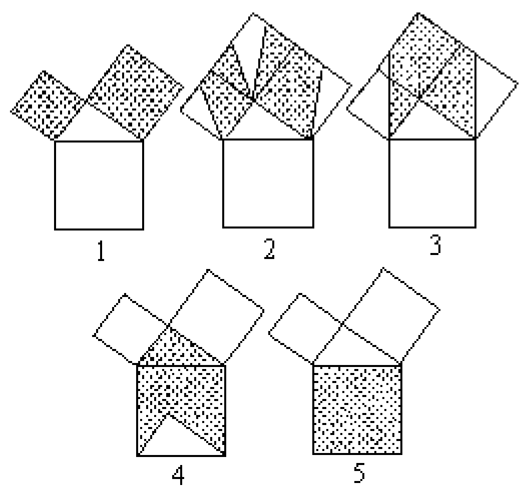


Draw a circle with radius c and a right triangle with sides a and b as shown. In this situation, one may apply any of a few well known facts. For example, in the diagram three points F, G, H located on the circle form another right triangle with the altitude FK of length a. Its hypotenuse GH is split in two pieces: (c + b) and (c - b). So, as in Proof #6, we get $a^2 = (c + b)(c - b) = c^2 - b^2$.

[Loomis, #53] attributes this construction to the great Leibniz, but lengthens the proof about threefold with meandering and misguided derivations.

B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v.3, n. 12 (1896), 299-300) offer a somewhat different route. Imagine FK is extended to the second intersection F' with the circle. Then, by the Intersecting Chords theorem, FK·KF' = GK·KH, with the same implication.

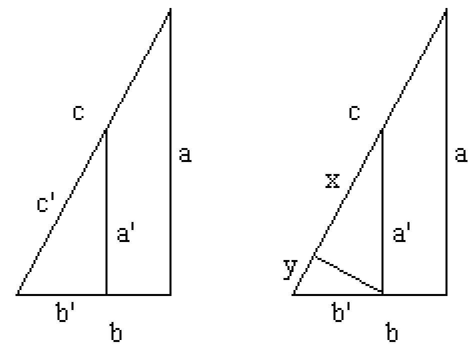
Proof #12



This proof is a variation on #1, one of the original Euclid's proofs. In parts 1,2, and 3, the two small squares are sheared towards each other such that the total shaded area remains unchanged (and equal to a^2+b^2.) In part 3, the length of the vertical portion of the shaded area's border is exactly c because the two leftover triangles are copies of the original one. This means one may slide down the shaded area as in part 4. From here the Pythagorean Theorem follows easily.

(This proof can be found in H. Eves, *In Mathematical Circles*, MAA, 2002, pp. 74-75)

Proof #13

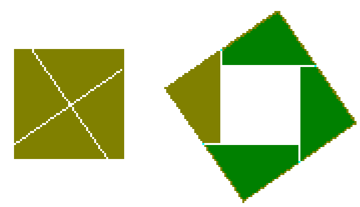


In the diagram there is several similar triangles (abc, a'b'c', a'x, and b'y.) We successively have

y/b = b'/c, x/a = a'/c, cy + cx = aa' + bb'.

And, finally, cc' = aa' + bb'. This is very much like Proof #6 but the result is more general.

Proof #14



This proof by H.E. Dudeney (1917) starts by cutting the square on the larger side into four parts that are then combined with the smaller one to form the square built on the hypotenuse.

Greg Frederickson from Purdue University, the author of a truly illuminating book, *Dissections: Plane & Fancy* (Cambridge University Press, 1997), pointed out the historical inaccuracy:

You attributed proof #14 to H.E. Dudeney (1917), but it was actually published earlier (1872) by Henry Perigal, a London stockbroker. A different dissection proof appeared much earlier, given by the Arabian mathematician/astronomer Thâbit in the tenth century. I have included details about these and other dissections proofs (including proofs of the Law of Cosines) in my recent book "Dissections: Plane & Fancy", Cambridge University Press, 1997. You might enjoy the web page for the book:

http://www.cs.purdue.edu/homes/gnf/book.html

Sincerely,

Greg Frederickson

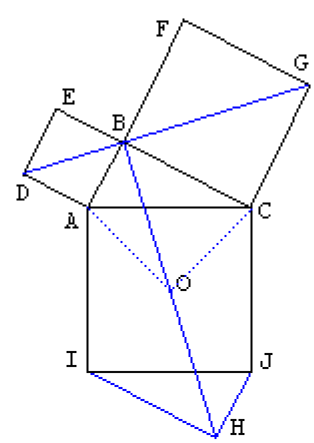
Bill Casselman from the University of British Columbia seconds Greg's information. Mine came from *Proofs Without Words* by R.B.Nelsen (MAA, 1993).

The proof has a dynamic version.

Proof #15

This remarkable proof by K. O. Friedrichs is a generalization of the previous one by Dudeney (or by Perigal, as above). It's indeed general. It's general in the sense that an infinite variety of specific geometric proofs may be derived from it. (Roger Nelsen ascribes [PWWII, p 3] this proof to Annairizi of Arabia (ca. 900 A.D.)) An especially nice variant by Olof Hanner appears on a separate page.

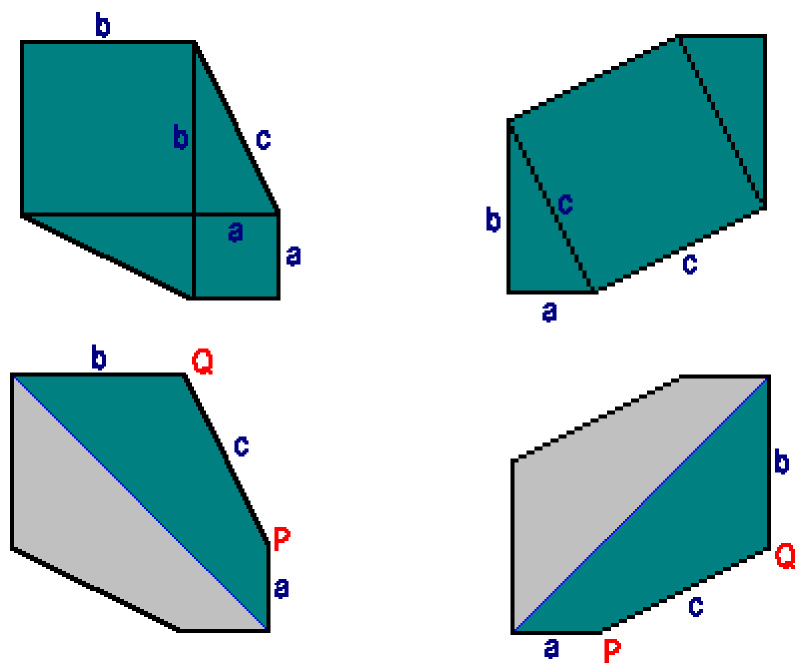
Proof #16



This proof is ascribed to Leonardo da Vinci (1452-1519) [Eves]. Quadrilaterals ABHI, JHBC, ADGC, and EDGF are all equal. (This follows from the observation that the angle ABH is 45°. This is so because ABC is right-angled, thus center O of the square ACJI lies on the circle circumscribing triangle ABC. Obviously, angle

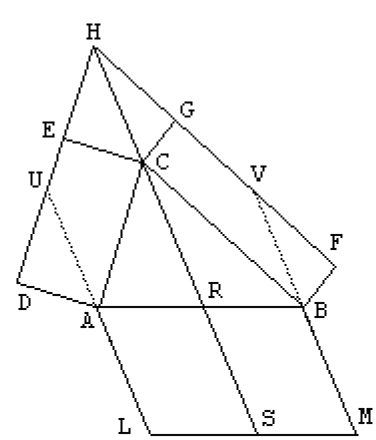
ABO is 45°.) Now, Area(ABHI) + Area(JHBC) = Area(ADGC) + Area(EDGF). Each sum contains two areas of triangles equal to ABC (IJH or BEF) removing which one obtains the Pythagorean Theorem.

David King modifies the argument somewhat



The side lengths of the hexagons are identical. The angles at P (right angle + angle between a & c) are identical. The angles at Q (right angle + angle between b & c) are identical. Therefore all four hexagons are identical.

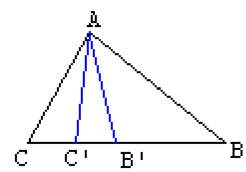
Proof #17



This proof appears in the Book IV of *Mathematical Collection* by Pappus of Alexandria (ca A.D. 300) [Eves, Pappas]. It generalizes the Pythagorean Theorem in two ways: the triangle ABC is not required to be right-angled and the shapes built on its sides are arbitrary parallelograms instead of squares. Thus build parallelograms CADE and CBFG on sides AC and, respectively, BC. Let DE and FG meet in H and draw AL and BM parallel and equal to HC. Then Area(ABML) = Area(CADE) + Area(CBFG). Indeed, with the sheering transformation already used in proofs #1 and #12, Area(CADE) = Area(CAUH) = Area(SLAR) and also Area(CBFG) = Area(CBVH) = Area(SMBR). Now, just add up what's equal.

A dynamic illustration is available elsewhere.

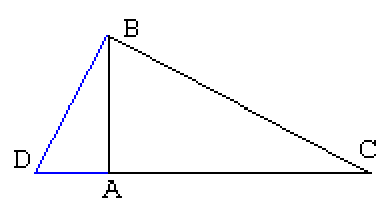
Proof #18



This is another generalization that does not require right angles. It's due to Thâbit ibn Qurra (836-901) [Eves]. If angles CAB, AC'B and AB'C are equal then $AC^2 + AB^2 = BC(CB' + BC')$. Indeed, triangles ABC, AC'B and AB'C are similar. Thus we have AB/BC' = BC/AB and AC/CB' = BC/AC which immediately leads to the required identity. In case the angle A is right, the theorem reduces to the Pythagorean proposition and proof #6.

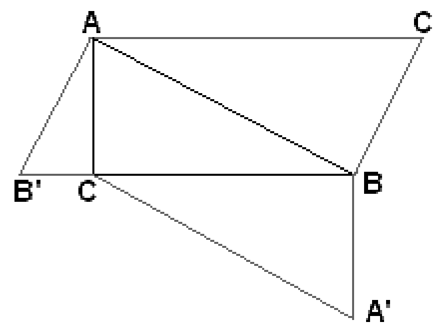
The same diagram is exploited in a different way by E. W. Dijkstra who concentrates on comparison of BC with the sum CB' + BC'.

Proof #19



This proof is a variation on #6. On the small side AB add a right-angled triangle ABD similar to ABC. Then, naturally, DBC is similar to the other two. From Area(ABD) + Area(ABC) = Area(DBC), $AD = AB^2/AC$ and $BD = AB\cdot BC/AC$ we derive $(AB^2/AC)\cdot AB + AB\cdot AC = (AB\cdot BC/AC)\cdot BC$. Dividing by AB/AC leads to $AB^2 + AC^2 = BC^2$.

Proof #20



This one is a cross between #7 and #19. Construct triangles ABC', BCA', and ACB' similar to ABC, as in the diagram. By construction, ΔABC = ΔA'BC. In addition, triangles ABB' and ABC' are also equal. Thus we conclude that Area(A'BC) + Area(AB'C) = Area(ABC'). From the similarity of triangles we get as before $B'C = AC^2/BC$ and $BC' = AC\cdot AB/BC$. Putting it all together yields $AC\cdot BC + (AC^2/BC)\cdot AC = AB\cdot(AC\cdot AB/BC)$ which is the same as

$$BC^2 + AC^2 = AB^2.$$

Proof #21

The following is an excerpt from a letter by Dr. Scott Brodie from the Mount Sinai School of Medicine, NY who sent me a couple of proofs of the theorem proper and its generalization to the Law of Cosines:

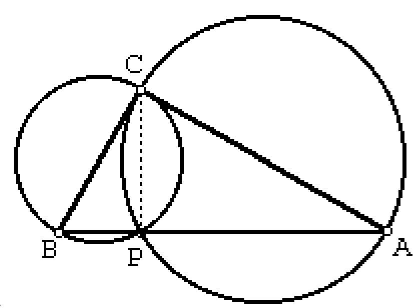
> The first proof I merely pass on from the excellent discussion in the Project Mathematics series, based on Ptolemy's theorem on quadrilaterals inscribed

in a circle: for such quadrilaterals, the sum of the products of the lengths of the opposite sides, taken in pairs equals the product of the lengths of the two diagonals. For the case of a rectangle, this reduces immediately to $a^2 + b^2 = c^2$.

Proof #22

Here is the second proof from Dr. Scott Brodie's letter.

We take as known a "power of the point" theorems: If a point is taken exterior to a circle, and from the point a segment is drawn tangent to the circle and another segment (a secant) is drawn which cuts the circle in two distinct points, then the square of the length of the tangent is equal to the product of the distance along the secant from the external point to the nearer point of intersection with the circle and the distance along the secant to the farther point of intersection with the circle.



Let ABC be a right triangle, with the right angle at C. Draw the altitude from C to the hypotenuse; let P denote the foot of this altitude. Then since CPB is right, the point P lies on the circle with diameter BC; and since CPA is right, the point P lies on the circle with diameter AC. Therefore the intersection of the two circles on the legs BC, CA of the original right triangle coincides with P, and in particular, lies on AB. Denote by x and y the lengths of segments BP and PA, respectively, and, as usual let a, b, c denote the lengths of the sides of ABC opposite the angles A, B, C respectively. Then, $x + y = c$.

Since angle C is right, BC is tangent to the circle with diameter CA, and the power theorem states that $a^2 = xc$; similarly, AC is tangent to the circle with diameter BC, and $b^2 = yc$. Adding, we find $a^2 + b^2 = xc + yc = c^2$, Q.E.D.

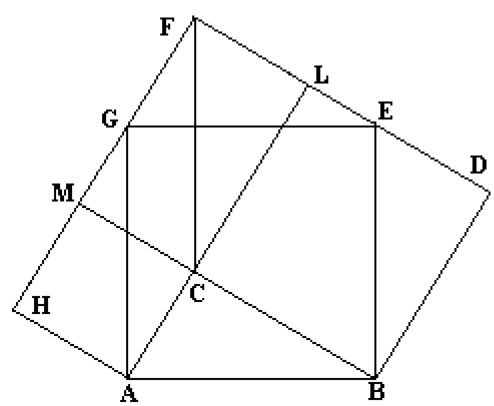
Dr. Brodie also created a Geometer's SketchPad file to illustrate this proof.

(This proof has been published as number XXIV in a collection of proofs by B. F. Yanney and J. A. Calderhead in *Am Math Monthly*, v. 4, n. 1 (1897), pp. 11-12.)

Proof #23

Another proof is based on the Heron's formula. (In passing, with the help of the formula I displayed the areas in the applet that illustrates Proof #7). This is a rather convoluted way to prove the Pythagorean Theorem that, nonetheless reflects on the centrality of the Theorem in the geometry of the plane. (A shorter and a more transparent application of Heron's formula is the basis of proof #75.)

Proof #24



[Swetz] ascribes this proof to abu' l'Hasan Thâbit ibn Qurra Marwân al'Harrani (826-901). It's the second of the proofs given by Thâbit ibn Qurra. The first one is essentially the #2 above.

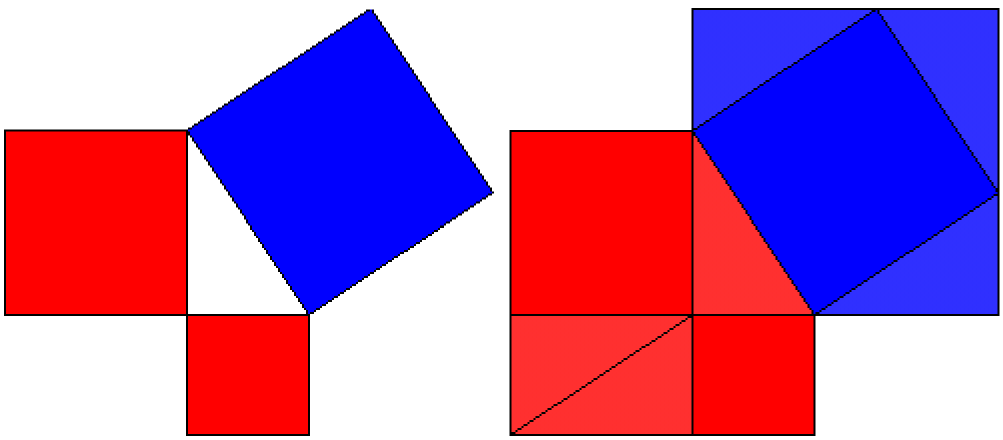
The proof resembles part 3 from proof #12. ΔABC = ΔFLC = ΔFMC = ΔBED = ΔAGH = ΔFGE. On one hand, the area of the shape ABDFH equals

$AC^2 + BC^2$ + Area(ΔABC + ΔFMC + ΔFLC). On the other hand,

Area(ABDFH) = AB^2 + Area(ΔBED + ΔFGE + ΔAGH).

Thâbit ibn Qurra's admits a natural generalization to a proof of the Law of Cosines.

A dynamic illustration of ibn Qurra's proof is also available.

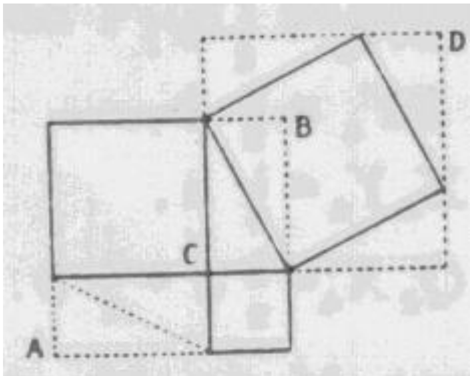


This is an "unfolded" variant of the above proof. Two pentagonal regions - the red and the blue - are obviously equal and leave the same area upon removal of three equal triangles from each.

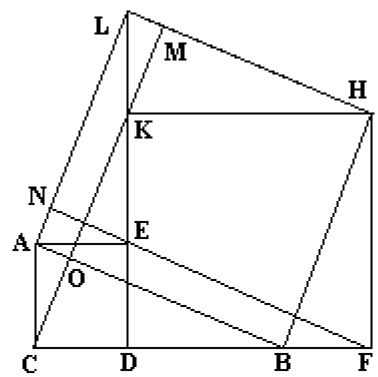
The proof is popularized by Monty Phister, author of the inimitable *Gnarly Math* CD-ROM.

Floor van Lamoen has gracefully pointed me to an earlier source. Eduard Douwes Dekker, one of the most famous Dutch authors, published in 1888 under the pseudonym of Multatuli a proof accompanied by the following diagram.

Scott Brodie pointed to the obvious relation of this proof to [# 9](). It is the same configuration but short of one triangle.

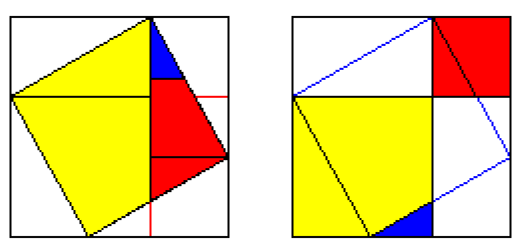


Proof #25



B.F.Yanney (1903, [Swetz]) gave a proof using the "shearing argument" also employed in the Proofs #1 and #12. Successively, areas of LMOA, LKCA, and ACDE (which is AC^2) are equal as are the areas of HMOB, HKCB, and HKDF (which is BC^2). BC = DF. Thus $AC^2 + BC^2$ = Area(LMOA) + Area(HMOB) = Area(ABHL) = AB^2.

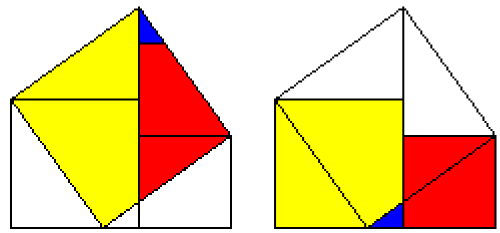
Proof #26



This proof I discovered at the site maintained by Bill Casselman where it is presented by a Java applet.

With all the above proofs, this one must be simple. Similar triangles like in proofs #6 or #13.

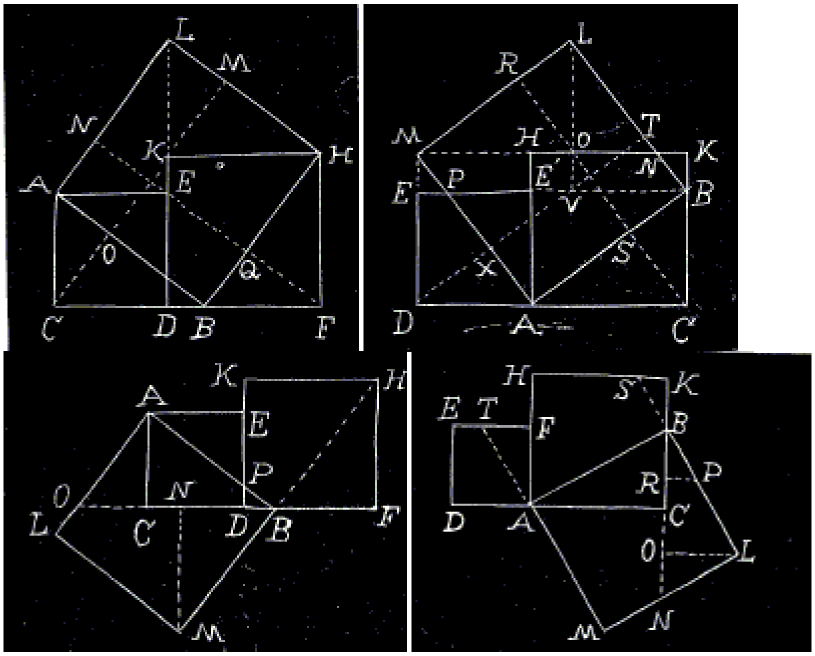
Proof #27



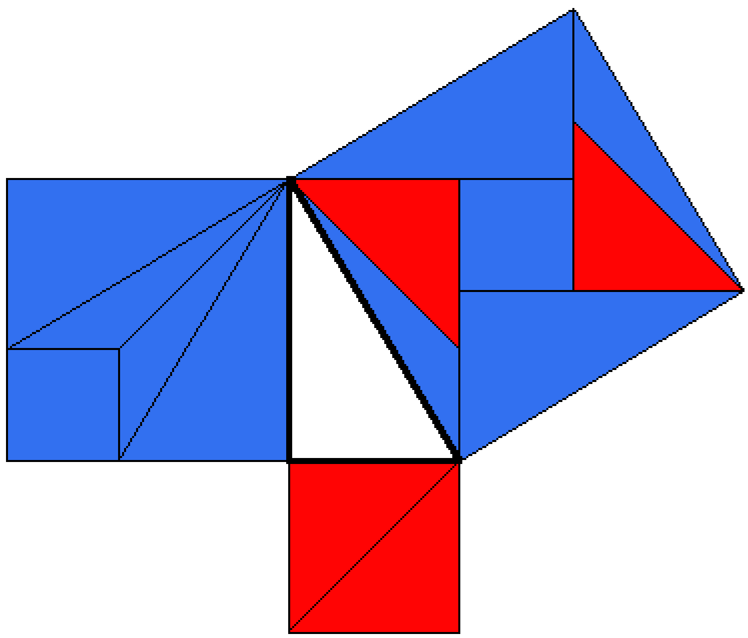
The same pieces as in proof #26 may be rearranged in yet another manner.

This dissection is often attributed to the 17th century Dutch mathematician Frans van Schooten. [Frederickson, p. 35] considers it as a hinged variant of one by ibn Qurra, see the note in parentheses following proof #2. Dr. France Dacar from Slovenia has pointed out that this same diagram is easily explained with a tessellation in proof #15. As a matter of fact, it may be better explained by a different tessellation. (I thank Douglas Rogers for setting this straight for me.)

The configuration at hand admits numerous variations. B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v. 6, n. 2 (1899), 33-34) published several proofs based on the following diagrams (multiple proofs per diagram at that)



Proof #28



Melissa Running from MathForum has kindly sent me a link (that since disappeared) to a page by Donald B. Wagner, an expert on history of science and technology in China. Dr. Wagner appeared to have reconstructed a proof by Liu Hui (third century AD). However (see below), there are serious doubts to the authorship of the proof.

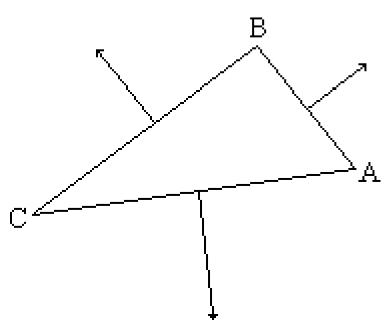
Elisha Loomis cites this as the geometric proof #28 with the following comment:

a. Benjir von Gutheil, oberlehrer at Nurnberg, Germany, produced the above proof. He died in the trenches in France, 1914. So wrote J. Adams, August 1933.
b. Let us call it the B. von Gutheil World War Proof.

Judging by the Sweet Land movie, such forgiving attitude towards a German colleague may not have been common at the time close to the WWI. It might have been even more guarded in the 1930s during the rise to power of the nazis in Germany.

(I thank D. Rogers for bringing the reference to Loomis' collection to my attention. He also expressed a reservation as regard the attribution of the proof to Liu Hui and traced its early appearance to Karl Julius Walther Lietzmann's *Geometrische aufgabensamming Ausgabe B: fuer Realanstalten*, published in Leipzig by Teubner in 1916. Interestingly, the proof has not been included in Lietzmann's earlier *Der Pythagoreische Lehrsatz* published in 1912.)

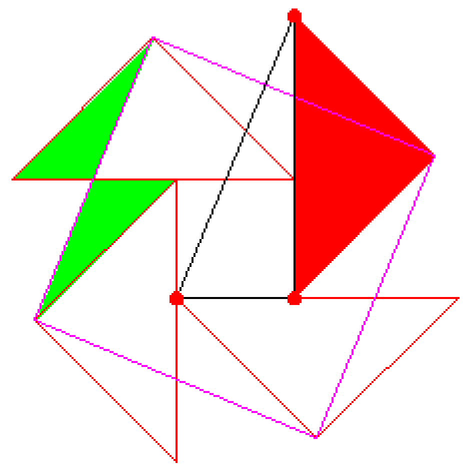
Proof #29



A mechanical proof of the theorem deserves a page of its own.

Pertinent to that proof is a page "Extra-geometric" proofs of the Pythagorean Theorem by Scott Brodie

Proof #30

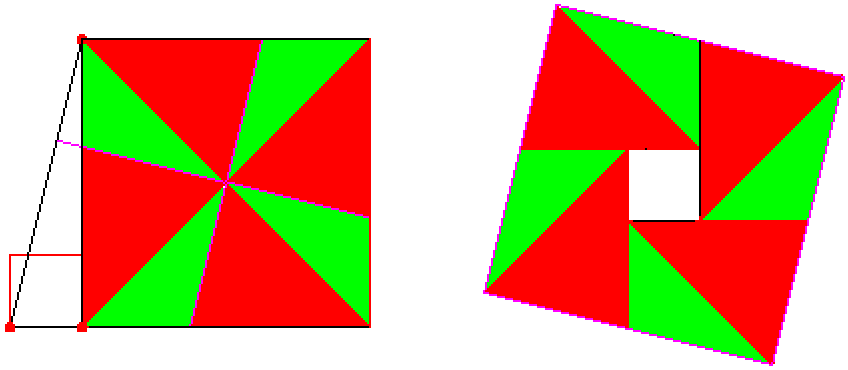


This proof I found in R. Nelsen's sequel *Proofs Without Words II*. (It's due to Poo-sung Park and was originally published in *Mathematics Magazine*, Dec 1999). Starting with one of the sides of a right triangle, construct 4 congruent right isosceles triangles with hypotenuses of any subsequent two perpendicular and apices away from the given triangle. The hypotenuse of the first of these triangles (in red in the diagram) should coincide with one of the sides.

The apices of the isosceles triangles form a square with the side equal to the hypotenuse of the given triangle. The hypotenuses of those triangles cut the sides of the square at their midpoints. So that there appear to be 4 pairs of equal triangles (one of the pairs is in green). One of the triangles in the pair is inside the square, the other is outside. Let the sides of the original triangle be a, b, c (hypotenuse). If the first isosceles triangle was built on side b, then each has area $b^2/4$. We obtain

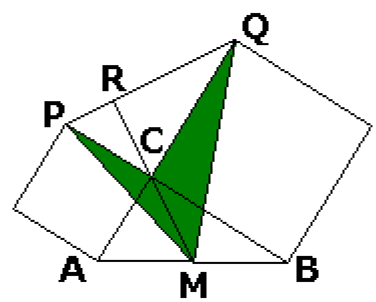
$$a^2 + 4b^2/4 = c^2$$

There's a dynamic illustration and another diagram that shows how to dissect two smaller squares and rearrange them into the big one.



This diagram also has a dynamic variant.

Proof #31



Given right ΔABC, let, as usual, denote the lengths of sides BC, AC and that of the hypotenuse as a, b, and c, respectively. Erect squares on sides BC and AC as on the diagram. According to SAS, triangles ABC and PCQ are equal, so that ∠QPC = ∠A. Let M be the midpoint of the hypotenuse. Denote the intersection of MC and PQ as R. Let's show that MR ⊥ PQ.

The median to the hypotenuse equals half of the latter. Therefore, ΔCMB is isosceles and ∠MBC = ∠MCB. But we also have ∠PCR = ∠MCB. From here and ∠QPC = ∠A it follows that angle CRP is right, or MR ⊥ PQ.

With these preliminaries we turn to triangles MCP and MCQ. We evaluate their areas in two different ways:

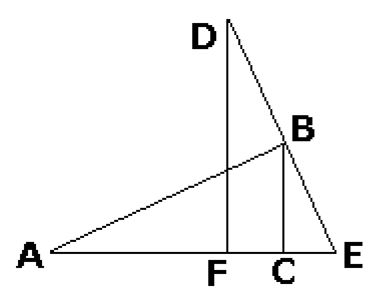
One one hand, the altitude from M to PC equals $AC/2 = b/2$. But also $PC = b$. Therefore, Area(ΔMCP) = $b^2/4$. On the other hand, Area(ΔMCP) = $CM\cdot PR/2 = c\cdot PR/4$. Similarly, Area($\Delta MCQ$) = $a^2/4$ and also Area(ΔMCQ) = $CM\cdot RQ/2 = c\cdot RQ/4$.

We may sum up the two identities: $a^2/4 + b^2/4 = c\cdot PR/4 + c\cdot RQ/4$, or $a^2/4 + b^2/4 = c\cdot c/4$.

(My gratitude goes to Floor van Lamoen who brought this proof to my attention. It appeared in *Pythagoras* - a dutch math magazine for schoolkids - in the December 1998 issue, in an article by Bruno Ernst. The proof is attributed to an American High School student from 1938 by the name of Ann Condit. The proof is included as the geometric proof 68 in Loomis' collection, p. 140.)

Proof #32

Let ABC and DEF be two congruent right triangles such that B lies on DE and A, F, C, E are collinear. $BC = EF = a$, $AC = DF = b$, $AB = DE = c$. Obviously, $AB \perp DE$. Compute the area of ΔADE in two different ways.



Area(ΔADE) = $AB\cdot DE/2 = c^2/2$ and also Area(ΔADE) = $DF\cdot AE/2 = b\cdot AE/2$. $AE = AC + CE = b + CE$. CE can be found from similar triangles BCE and DFE: $CE = BC\cdot FE/DF = a\cdot a/b$. Putting things together we obtain

$$c^2/2 = b(b + a^2/b)/2$$

(This proof is a simplification of one of the proofs by Michelle Watkins, a student at the University of North Florida, that appeared in *Math Spectrum* 1997/98, v30, n3, 53-54.)

Douglas Rogers observed that the same diagram can be treated differently:

Proof 32 can be tidied up a bit further, along the lines of the later proofs added more recently, and so avoiding similar triangles.

Of course, ADE is a triangle on base DE with height AB, so of area cc/2.

But it can be dissected into the triangle FEB and the quadrilateral ADBF. The former has base FE and height BC, so area aa/2. The latter in turn consists of two triangles back to back on base DF with combined heights AC, so area bb/2. An alternative dissection sees triangle ADE as consisting of triangle ADC and triangle CDE, which, in turn, consists of two triangles back to back on base BC, with combined heights EF.

The next two proofs have accompanied the following message from Shai Simonson, Professor at Stonehill College in Cambridge, MA:

Greetings,

I was enjoying looking through your site, and stumbled on the long list of Pyth Theorem Proofs.

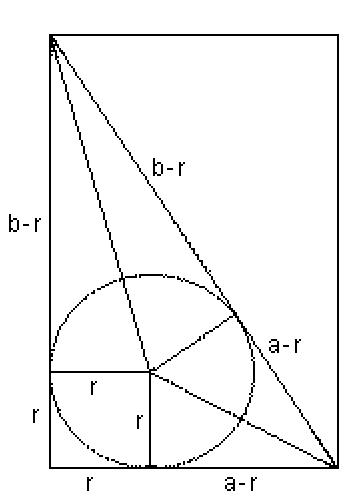
In my course "The History of Mathematical Ingenuity" I use two proofs that use an inscribed circle in a right triangle. Each proof uses two diagrams, and each is a different geometric view of a single algebraic proof that I discovered many years ago and published in a letter to Mathematics Teacher.

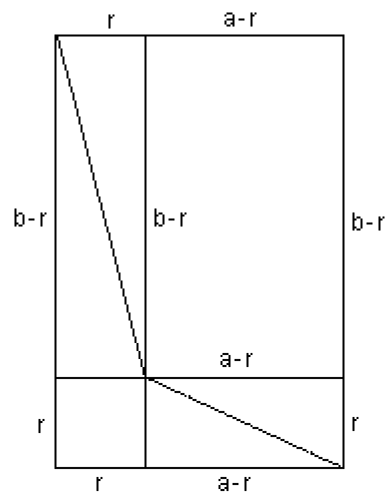
The two geometric proofs require no words, but do require a little thought.

Best wishes,

Shai

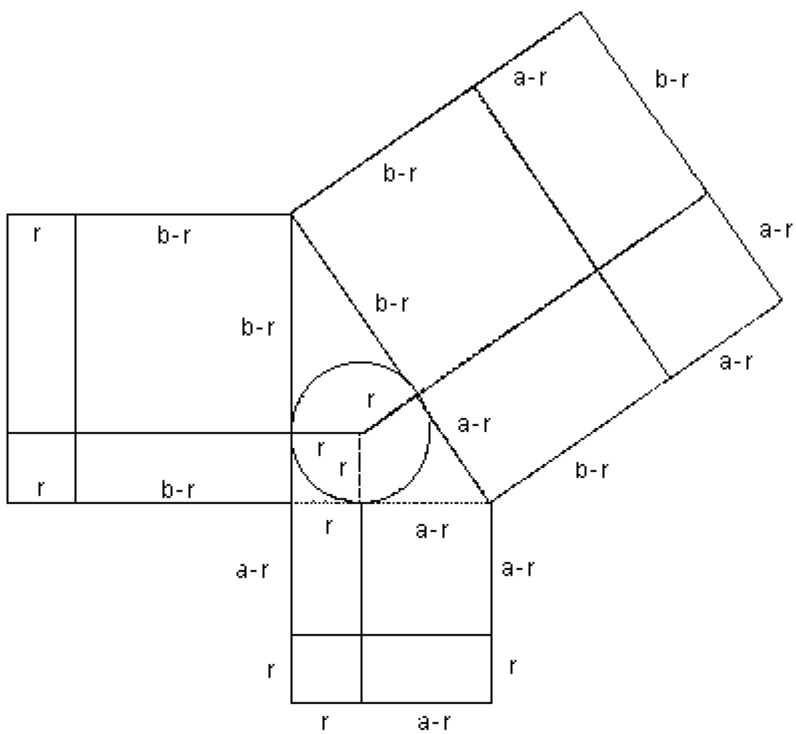
Proof #33





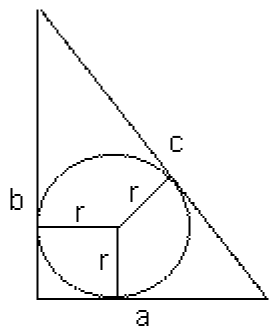
$$r^2 + r(a - r) + r(b - r) = ab/2$$

$$r(a + b - r) = ab/2 \qquad\qquad (a - r)(b - r) = ab/2$$

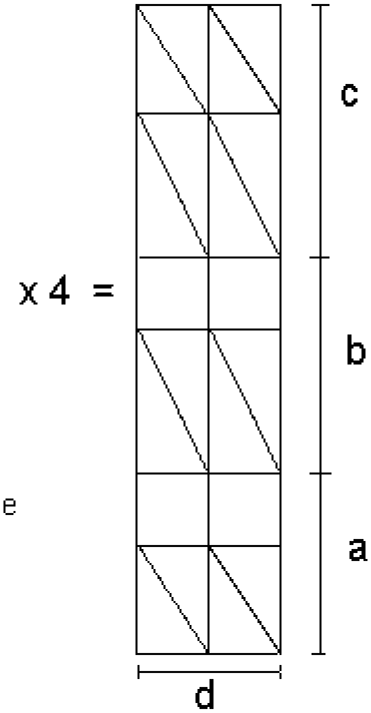


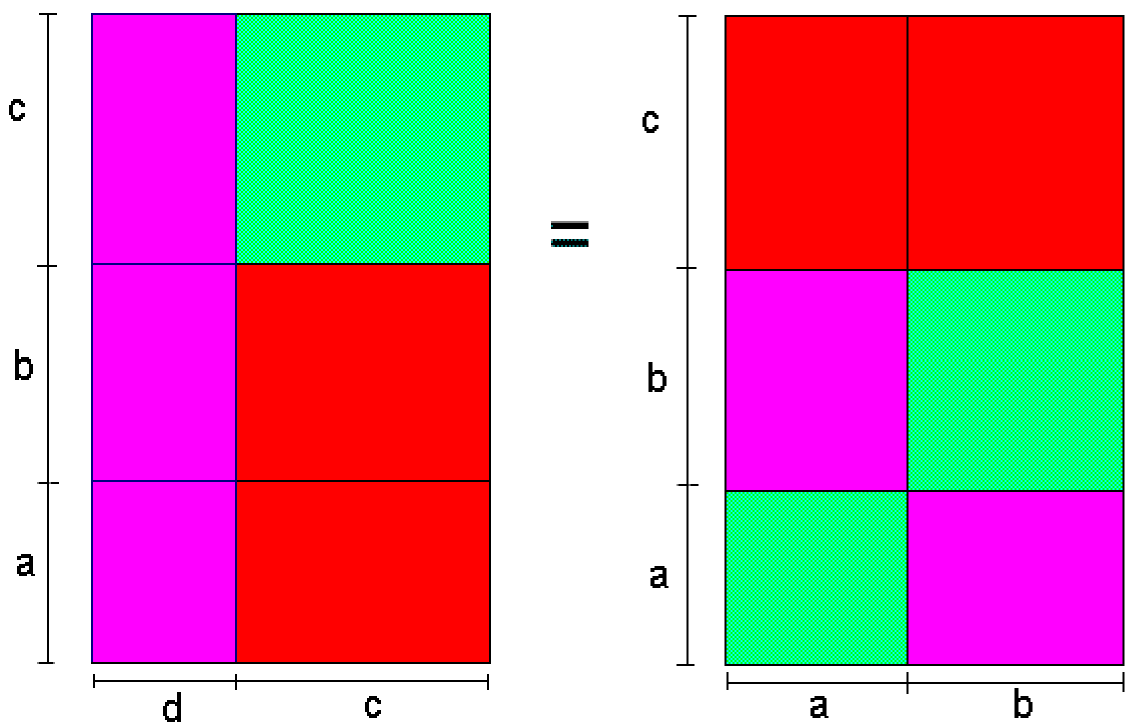
Proof #34

Pythagorean Theorem
Dissection using inscribed circle



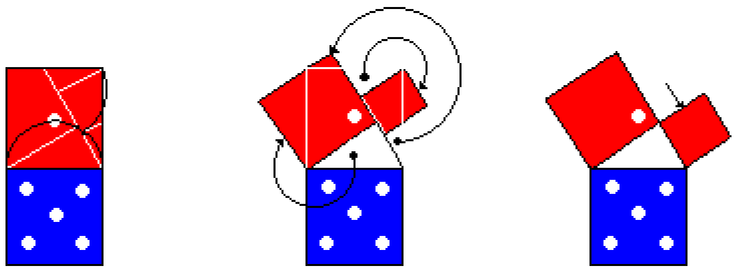
d = 2r = Diameter of a circle
inscribed in a right triangle





Proof #35

Cracked Domino - a proof by Mario Pacek (aka Pakoslaw Gwizdalski) - also requires some thought.



A B C

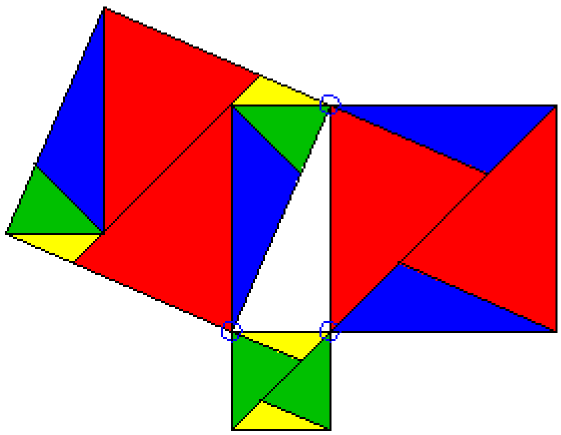
The proof sent via email was accompanied by the following message:

> This new, extraordinary and extremely elegant proof of quite probably the most fundamental theorem in mathematics (hands down winner with respect to the # of proofs 367?) is superior to all known to science including the Chinese and James A. Garfield's (20th US president), because it is direct, does not involve any formulas and even preschoolers can get it. Quite probably it is identical to the lost original one - but who can prove that? Not in the Guinness Book of Records yet!

The manner in which the pieces are combined may well be original. The dissection itself is well known (see Proofs 26 and 27) and is described in Frederickson's book, p. 29. It's remarked there that B. Brodie (1884) observed that the dissection like that also applies to similar rectangles. The dissection is also a particular instance of the superposition proof by K.O.Friedrichs.

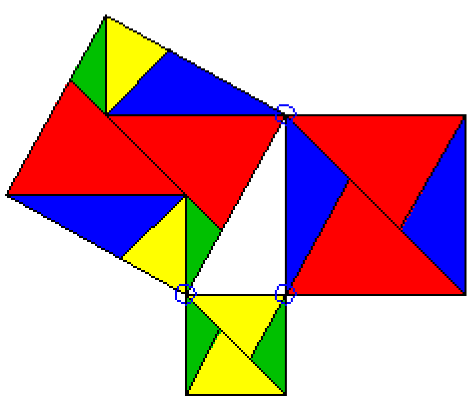
Proof #36

This proof is due to J. E. Böttcher and has been quoted by Nelsen (*Proofs Without Words II*, p. 6).



I think cracking this proof without words is a good exercise for middle or high school geometry class.

S. K. Stein, (*Mathematics: The Man-Made Universe*, Dover, 1999, p. 74) gives a slightly different dissection.



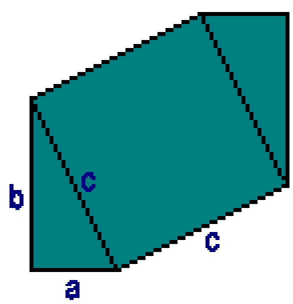
Both variants have a dynamic version.

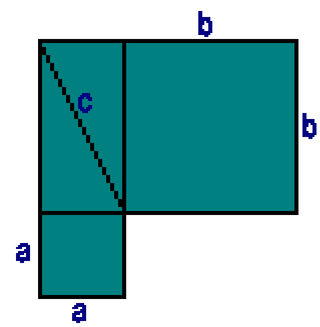
Proof #37

An applet by David King that demonstrates this proof has been placed on a [separate page](#).

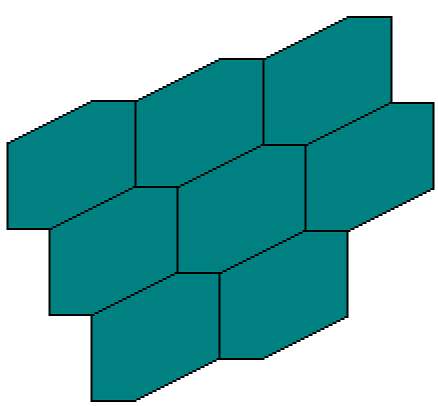
Proof #38

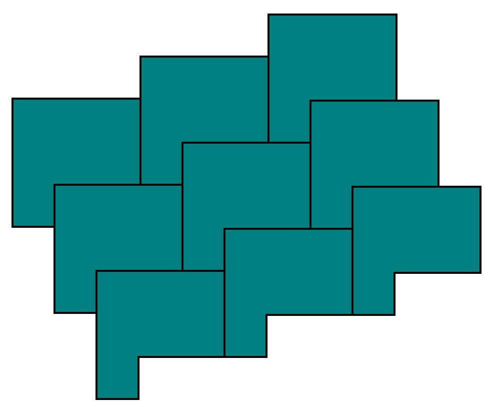
This proof was also communicated to me by David King. Squares and 2 triangles combine to produce two hexagon of equal area, which might have been established as in Proof #9. However, both hexagons tessellate the plane.





Both hexagons tessellate:



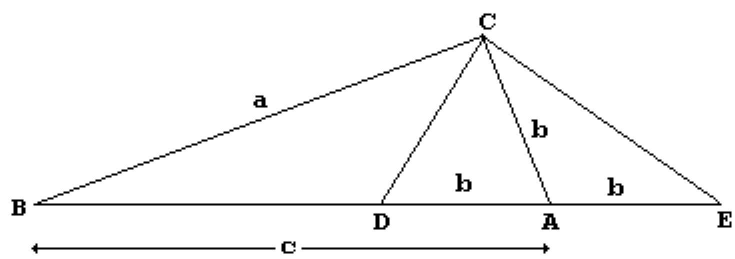


For every hexagon in the left tessellation there is a hexagon in the right tessellation. Both tessellations have the same lattice structure which is [demonstrated by an](#)

applet. The Pythagorean theorem is proven after two triangles are removed from each of the hexagons.

Proof #39

(By J. Barry Sutton, *The Math Gazette*, v 86, n 505, March 2002, p72.)



Let in ΔABC, angle C = 90°. As usual, AB = c, AC = b, BC = a. Define points D and E on AB so that AD = AE = b.

By construction, C lies on the circle with center A and radius b. Angle DCE subtends its diameter and thus is right: ∡DCE = 90°. It follows that ∡BCD = ∡ACE. Since ΔACE is isosceles, ∡CEA = ∡ACE.

Triangles DBC and EBC share ∡DBC. In addition, ∡BCD = ∡BEC. Therefore, triangles DBC and EBC are similar. We have BC/BE = BD/BC, or

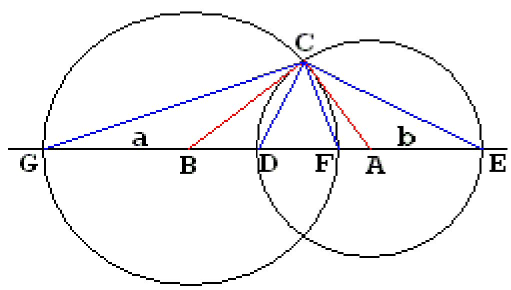
$$a / (c + b) = (c - b) / a.$$

And finally

$$a^2 = c^2 - b^2,$$

$$a^2 + b^2 = c^2.$$

The diagram reminds one of Thâbit ibn Qurra's proof. But the two are quite different. However, this is exactly proof 14 from Elisha Loomis' collection. Furthermore, Loomis provides two earlier references from 1925 and 1905. With the circle centered at A drawn, Loomis repeats the proof as 82 (with references from 1887, 1880, 1859, 1792) and also lists (as proof 89) a symmetric version of the above:



For the right triangle ABC, with right angle at C, extend AB in both directions so that AE = AC = b and BG = BC = a. As above we now have triangles DBC and EBC similar. In addition, triangles AFC and ACG are also similar, which results in two identities:

$a^2 = c^2 - b^2$, and

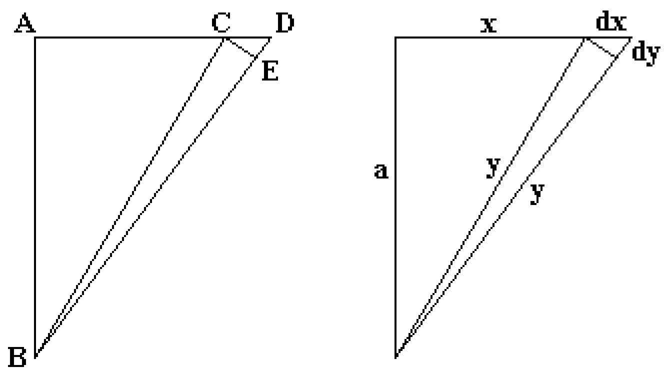
$b^2 = c^2 - a^2$.

Instead of using either of the identities directly, Loomis adds the two:

$2(a^2 + b^2) = 2c^2$,

which appears as both graphical and algebraic overkill.

Proof #40



This one is by Michael Hardy from University of Toledo and was published in *The Mathematical Intelligencer* in 1988. It must be taken with a grain of salt.

Let ABC be a right triangle with hypotenuse BC. Denote AC = x and BC = y. Then, as C moves along the line AC, x changes and so does y. Assume x changed by a small amount dx. Then y changed by a small amount dy. The triangle CDE may be approximately considered right. Assuming it is, it shares one angle (D) with triangle ABD, and is therefore similar to the latter. This leads to the proportion $x/y = dy/dx$, or a (separable) differential equation

$$y\cdot dy - x\cdot dx = 0,$$

which after integration gives $y^2 - x^2 = \text{const}$. The value of the constant is determined from the initial condition for $x = 0$. Since $y(0) = a$, $y^2 = x^2 + a^2$ for all x.

It is easy to take an issue with this proof. What does it mean for a triangle to be approximately right? I can offer the following explanation. Triangles ABC and ABD are right by construction. We have, $AB^2 + AC^2 = BC^2$ and also $AB^2 + AD^2 = BD^2$, by the Pythagorean theorem. In terms of x and y, the theorem appears as

$$x^2 + a^2 = y^2$$

$$(x + dx)^2 + a^2 = (y + dy)^2$$

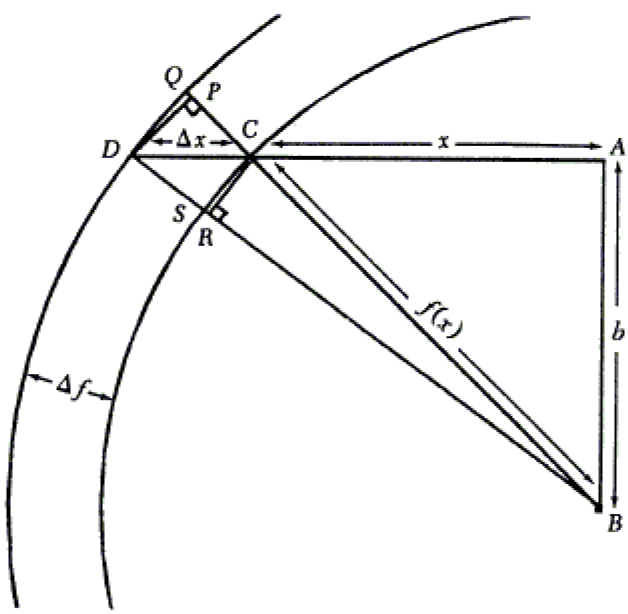
which, after subtraction, gives

$$y \cdot dy - x \cdot dx = (dx^2 - dy^2)/2.$$

For small dx and dy, dx^2 and dy^2 are even smaller and might be neglected, leading to the approximate $y \cdot dy - x \cdot dx = 0$. ■

The trick in Michael's vignette is in skipping the issue of approximation. But can one really justify the derivation without relying on the Pythagorean theorem in the first place? Regardless, I find it very much to my enjoyment to have the ubiquitous equation $y \cdot dy - x \cdot dx = 0$ placed in that geometric context.

An amplified, but apparently independent, version of this proof has been published by Mike Staring (*Mathematics Magazine*, V. 69, n. 1 (Feb., 1996), 45-46).



Assuming $\Delta x > 0$ and detecting similar triangles,

$$\Delta f / \Delta x = CQ/CD > CP/CD = CA/CB = x/f(x).$$

But also,

$$\Delta f / \Delta x = SD/CD < RD/CD = AD/BD = (x + \Delta x) / (f(x) + \Delta f) < x/f(x) + \Delta x/f(x).$$

Passing to the limit as Δx tends to 0^+, we get

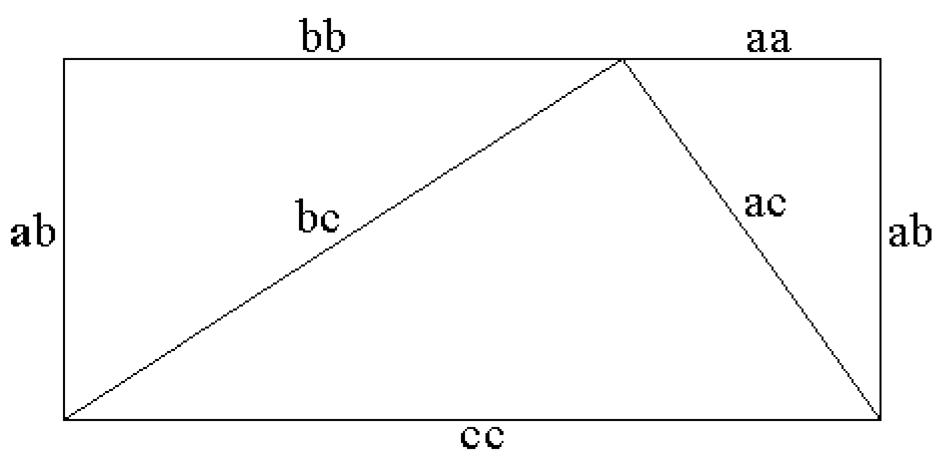
$$df / dx = x / f(x).$$

The case of $\Delta x < 0$ is treated similarly. Now, solving the differential equation we get

$$f^2(x) = x^2 + c.$$

The constant c is found from the boundary condition $f(0) = b$: $c = b^2$. And the proof is complete.

Proof #41

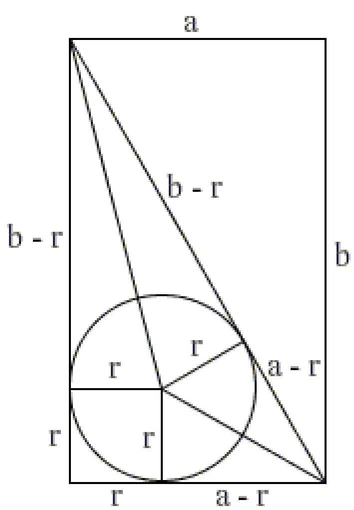


Create 3 scaled copies of the triangle with sides a, b, c by multiplying it by a, b, and c in turn. Put together, the three similar triangles thus obtained to form a rectangle whose upper side is $a^2 + b^2$, whereas the lower side is c^2.

For additional details and modifications see a separate page.

Proof #42

The proof is based on the same diagram as #33 [Pritchard, p. 226-227].



Area of a triangle is obviously rp, where r is the inradius and p = (a + b + c)/2 the semiperimeter of the triangle. From the diagram, the hypothenuse c = (a - r) + (b - r), or r = p - c. The area of the triangle then is computed in two ways:

$$p(p - c) = ab/2,$$

which is equivalent to

$$(a + b + c)(a + b - c) = 2ab,$$

or

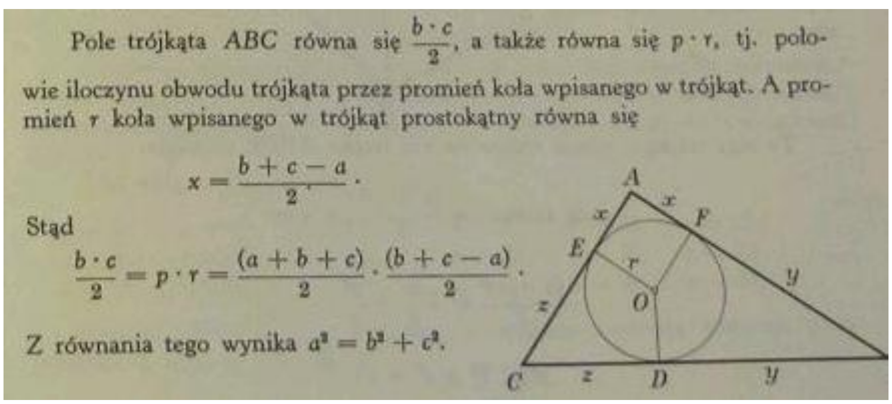
$$(a + b)^2 - c^2 = 2ab.$$

And finally

$$a^2 + b^2 - c^2 = 0.$$

The proof is due to Jack Oliver, and was originally published in *Mathematical Gazette* **81** (March 1997), p 117-118.

Maciej Maderek informed me that the same proof appeared in a Polish 1988 edition of *Sladami Pitagorasa* by Szczepan Jelenski:


Pole trójkąta ABC równa się $\frac{b \cdot c}{2}$, a także równa się $p \cdot r$, tj. połowie iloczynu obwodu trójkąta przez promień koła wpisanego w trójkąt. A promień r koła wpisanego w trójkąt prostokątny równa się

$$x = \frac{b + c - a}{2}.$$

Stąd

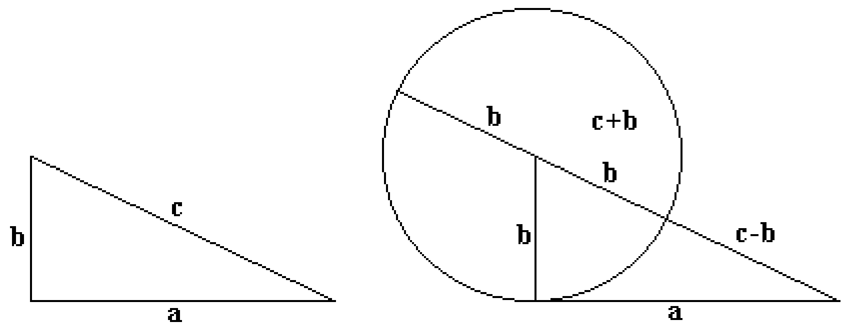
$$\frac{b \cdot c}{2} = p \cdot r = \frac{(a + b + c)}{2} \cdot \frac{(b + c - a)}{2}.$$

Z równania tego wynika $a^2 = b^2 + c^2$.

Jelenski attributes the proof to Möllmann without mentioning a source or a date.

Proof #43

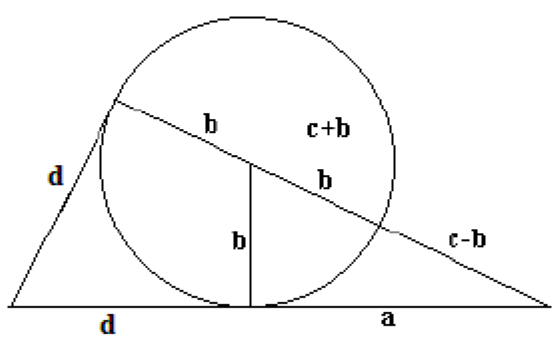
By Larry Hoehn [Pritchard, p. 229, and Math Gazette].



Apply the Power of a Point theorem to the diagram above where the side a serves as a tangent to a circle of radius b: $(c - b)(c + b) = a^2$. The result follows immediately.

(The configuration here is essentially the same as in proof #39. The invocation of the Power of a Point theorem may be regarded as a shortcut to the argument in proof #39. Also, this is exactly proof XVI by B. F. Yanney and J. A. Calderhead, *Am Math Monthly*, v.3, n. 12 (1896), 299-300.)

John Molokach suggested a modification based on the following diagram:



From the similarity of triangles, $a/b = (b + c)/d$, so that $d = b(b + c)/a$. The quadrilateral on the left is a kite with sides b and d and area $2bd/2 = bd$. Adding to this the area of the small triangle $(ab/2)$ we obtain the area of the big triangle - $(b + c)d/2$:

$bd + ab/2 = (b + c)d/2$

which simplifies to

$ab/2 = (c - b)d/2$, or $ab = (c - b)d$.

Now using the formula for d:

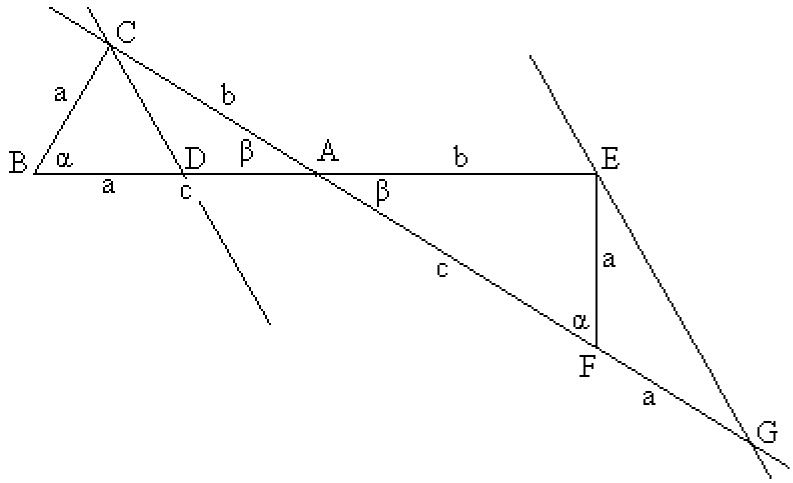
$ab = (c - b)d = (c - b)(c + b)b/a$.

Dividing by b and multiplying by a gives $a^2 = c^2 - b^2$. This variant comes very close to Proof #82, but with a different motivation.

Finally, the argument shows that the area of an annulus (ring) bounded by circles of radii b and $c > b$; is exactly πa^2 where $a^2 = c^2 - b^2$. a is a half length of the tangent to the inner circle enclosed within the outer circle.

Proof #44

The following proof related to #39, have been submitted by Adam Rose (Sept. 23, 2004.)



Start with two identical right triangles: ABC and AFE, A the intersection of BE and CF. Mark D on AB and G on extension of AF, such that

$$BC = BD = FG\ (= EF).$$

(For further notations refer to the above diagram.) ΔBCD is isosceles. Therefore, $\angle BCD = \pi/2 - \alpha/2$. Since angle C is right,

$$\angle ACD = \pi/2 - (\pi/2 - \alpha/2) = \alpha/2.$$

Since $\angle AFE$ is exterior to ΔEFG, $\angle AFE = \angle FEG + \angle FGE$. But ΔEFG is also isosceles. Thus

$$\angle AGE = \angle FGE = \alpha/2.$$

We now have two lines, CD and EG, crossed by CG with two *alternate interior* angles, ACD and AGE, equal. Therefore, CD||EG. Triangles ACD and AGE are similar, and $AD/AC = AE/AG$:

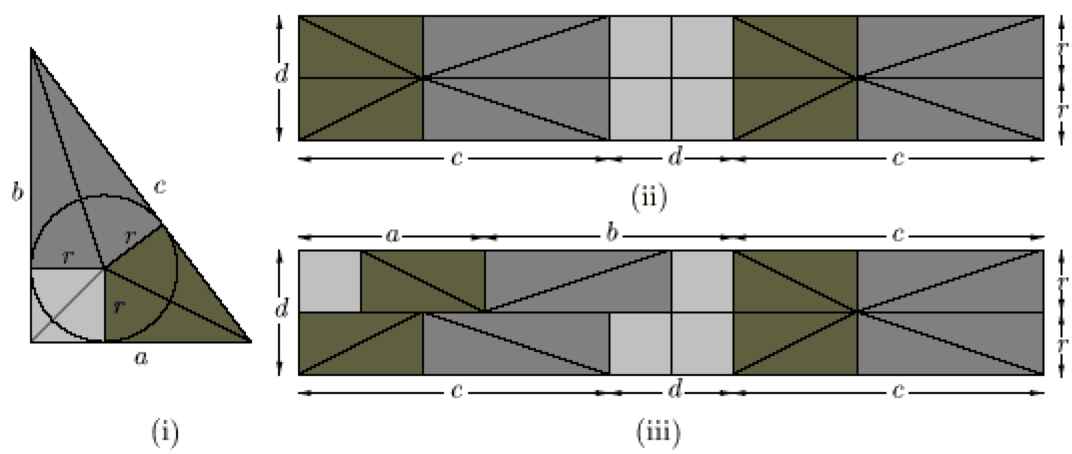
$$b/(c - a) = (c + a)/b,$$

and the Pythagorean theorem follows.

Proof #45

This proof is due to Douglas Rogers who came upon it in the course of his investigation into the history of Chinese mathematics.

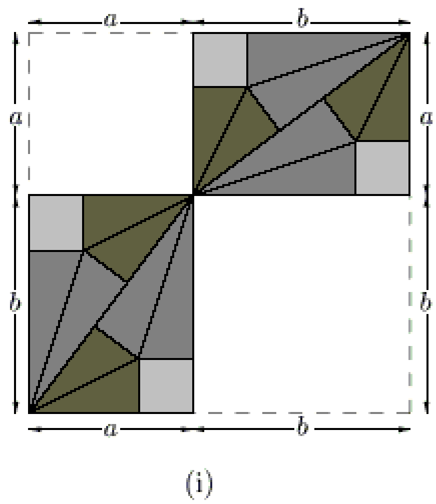
The proof is a variation on #33, #34, and #42. The proof proceeds in two steps. First, as it may be observed from

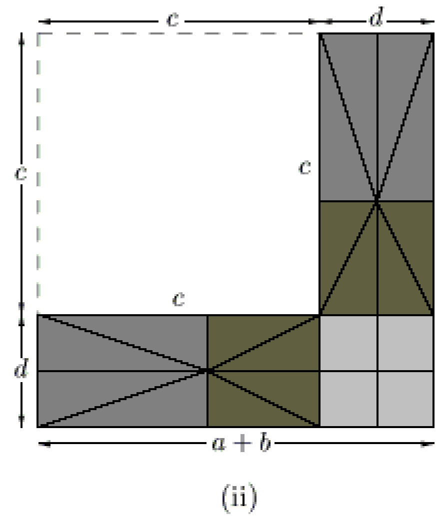


a Liu Hui identity (see also Mathematics in China)

$$a + b = c + d,$$

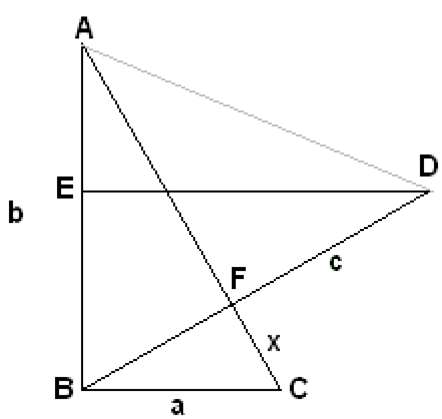
where d is the diameter of the circle inscribed into a right triangle with sides a and b and hypotenuse c. Based on that and rearranging the pieces in two ways supplies another proof without words of the Pythagorean theorem:





Proof #46

This proof is due to Tao Tong (*Mathematics Teacher*, Feb., 1994, Reader Reflections). I learned of it through the good services of Douglas Rogers who also brought to my attention Proofs #47, #48 and #49. In spirit, the proof resembles the proof #32.



Let ABC and BED be equal right triangles, with E on AB. We are going to evaluate the area of ΔABD in two ways:

Area(ΔABD) = BD·AF/2 = DE·AB/2.

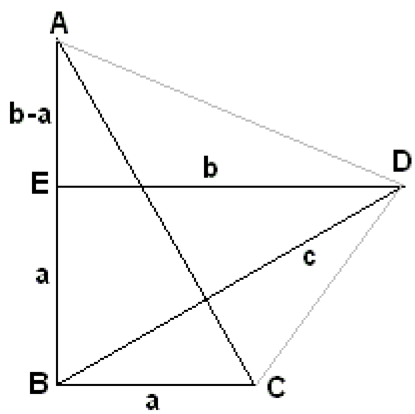
Using the notations as indicated in the diagram we get $c(c - x)/2 = b \cdot b/2$. $x = CF$ can be found by noting the similarity ($BD \perp AC$) of triangles BFC and ABC:

$$x = a^2/c.$$

The two formulas easily combine into the Pythagorean identity.

Proof #47

This proof which is due to a high school student John Kawamura was report by Chris Davis, his geometry teacher at Head-Rouce School, Oakland, CA *(Mathematics Teacher*, Apr., 2005, p. 518.)



The configuration is virtually identical to that of Proof #46, but this time we are interested in the area of the quadrilateral ABCD. Both of its perpendicular diagonals have length c, so that its area equals $c^2/2$. On the other hand,

$$c^2/2 = \text{Area}(ABCD)$$

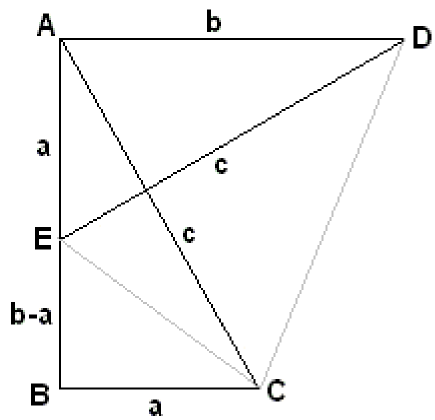
$$= \text{Area}(BCD) + \text{Area}(ABD)$$

= a·a/2 + b·b/2

Multiplying by 2 yields the desired result.

Proof #48

(W. J. Dobbs, *The Mathematical Gazette*, 8 (1915-1916), p. 268.)



In the diagram, two right triangles - ABC and ADE - are equal and E is located on AB. As in President Garfield's proof, we evaluate the area of a trapezoid ABCD in two ways:

Area(ABCD) = Area(AECD) + Area(BCE)

= c·c/2 + a(b - a)/2,

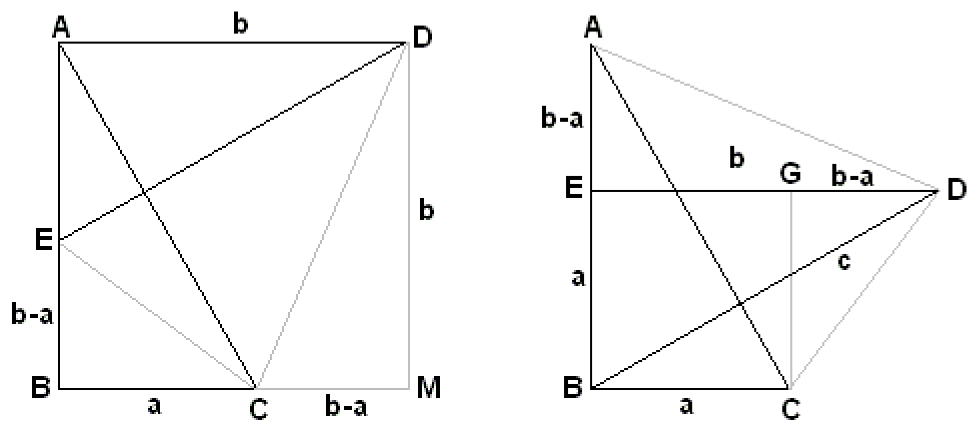
where, as in the proof #47, c·c is the product of the two perpendicular diagonals of the quadrilateral AECD. On the other hand,

Area(ABCD) = AB·(BC + AD)/2

$= b(a + b)/2.$

Combining the two we get $c^2/2 = a^2/2 + b^2/2$, or, after multiplication by 2, $c^2 = a^2 + b^2$.

Proof #49



In the previous proof we may proceed a little differently. Complete a square on sides AB and AD of the two triangles. Its area is, on one hand, b^2 and, on the other,

b^2 = Area(ABMD)

= Area(AECD) + Area(CMD) + Area(BCE)

$= c^2/2 + b(b - a)/2 + a(b - a)/2$

$= c^2/2 + b^2/2 - a^2/2,$

which amounts to the same identity as before.

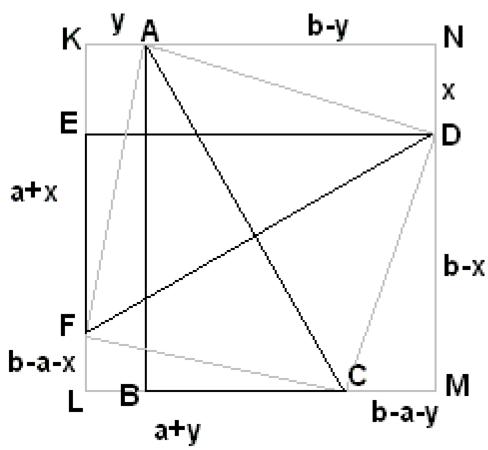
Douglas Rogers who observed the relationship between the proofs 46-49 also remarked that a square could have been drawn on the smaller legs of the two triangles if the second triangle is drawn in the "bottom" position as in proofs 46 and 47. In this case, we will again evaluate the area of the quadrilateral ABCD in two ways. With a reference to the second of the diagrams above,

$$
\begin{aligned}
c^2/2 &= \text{Area}(ABCD)\\
&= \text{Area}(EBCG) + \text{Area}(CDG) + \text{Area}(AED)\\
&= a^2 + a(b - a)/2 + b(b - a)/2\\
&= a^2/2 + b^2/2,
\end{aligned}
$$

as was desired.

He also pointed out that it is possible to think of one of the right triangles as sliding from its position in proof #46 to its position in proof #48 so that its short leg glides along the long leg of the other triangle. At any intermediate position there is present a quadrilateral with equal and perpendicular diagonals, so that for all positions it is possible to construct proofs analogous to the above. The triangle always remains inside a square of side b - the length of the long leg of the two triangles. Now, we can also imagine the triangle ABC slide inside that square. Which leads to a proof that directly generalizes #49 and includes configurations of proofs 46-48. See below.

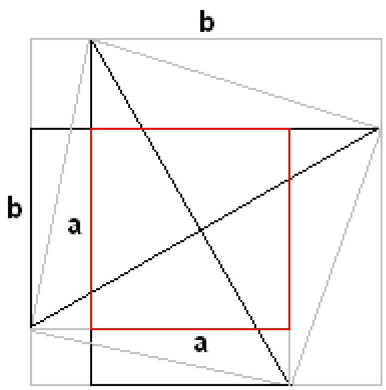
Proof #50



The area of the big square KLMN is b^2. The square is split into 4 triangles and one quadrilateral:

$$
\begin{aligned}
b^2 &= \text{Area}(KLMN) \\
&= \text{Area}(AKF) + \text{Area}(FLC) + \text{Area}(CMD) + \text{Area}(DNA) + \text{Area}(AFCD) \\
&= y(a+x)/2 + (b-a-x)(a+y)/2 + (b-a-y)(b-x)/2 + x(b-y)/2 + c^2/2 \\
&= [y(a+x) + b(a+y) - y(a+x) - x(b-y) - a\cdot a + (b-a-y)b + x(b-y) + c^2]/2 \\
&= [b(a+y) - a\cdot a + b\cdot b - (a+y)b + c^2]/2 \\
&= b^2/2 - a^2/2 + c^2/2.
\end{aligned}
$$

It's not an interesting derivation, but it shows that, when confronted with a task of simplifying algebraic expressions, multiplying through all terms as to remove all parentheses may not be the best strategy. In this case, however, there is even a better strategy that avoids lengthy computations altogether. On Douglas Rogers' suggestion, complete each of the four triangles to an appropriate rectangle:

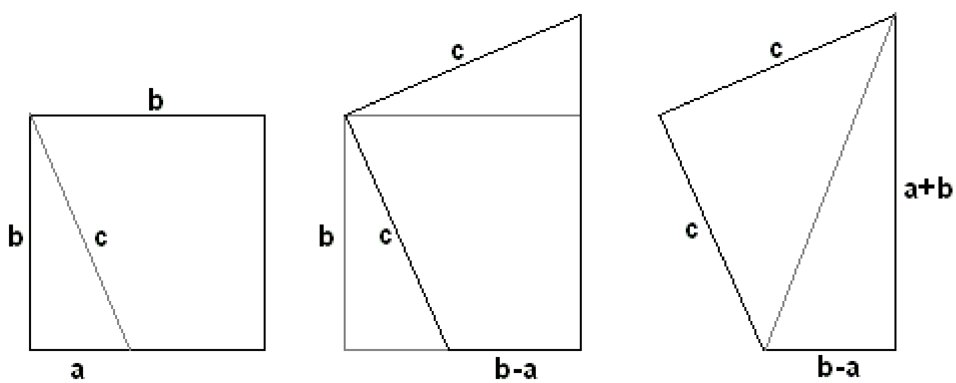


The four rectangles always cut off a square of size a, so that their total area is $b^2 - a^2$. Thus we can finish the proof as in the other proofs of this series:

$$b^2 = c^2/2 + (b^2 - a^2)/2.$$

Proof #51

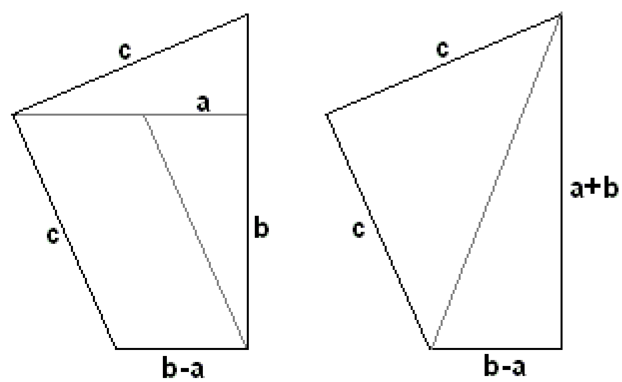
(W. J. Dobbs, *The Mathematical Gazette*, 7 (1913-1914), p. 168.)



This one comes courtesy of Douglas Rogers from his extensive collection. As in Proof #2, the triangle is rotated 90 degrees around one of its corners, such that the angle between the hypotenuses in two positions is right. The resulting shape of area b^2 is then dissected into two right triangles with side lengths (c, c) and (b - a, a + b) and areas $c^2/2$ and $(b - a)(a + b)/2 = (b^2 - a^2)/2$:

$$b^2 = c^2/2 + (b^2 - a^2)/2.$$

J. Elliott adds a wrinkle to the proof by turning around one of the triangles:



Again, the area can be computed in two ways:

$$ab/2 + ab/2 + b(b - a) = c^2/2 + (b - a)(b + a)/2,$$

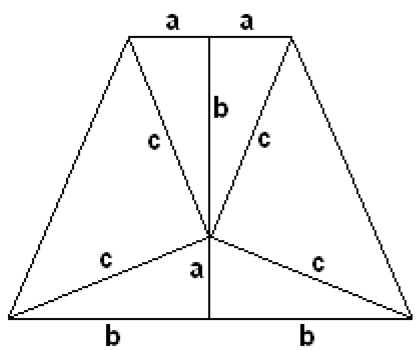
which reduces to

$$b^2 = c^2/2 + (b^2 - a^2)/2,$$

and ultimately to the Pythagorean identity.

Proof #52

This proof, discovered by a high school student, Jamie deLemos (*The Mathematics Teacher*, 88 (1995), p. 79.), has been quoted by Larry Hoehn (*The Mathematics Teacher*, 90 (1997), pp. 438-441.)



On one hand, the area of the trapezoid equals

$$(2a + 2b)/2\cdot(a + b)$$

and on the other,

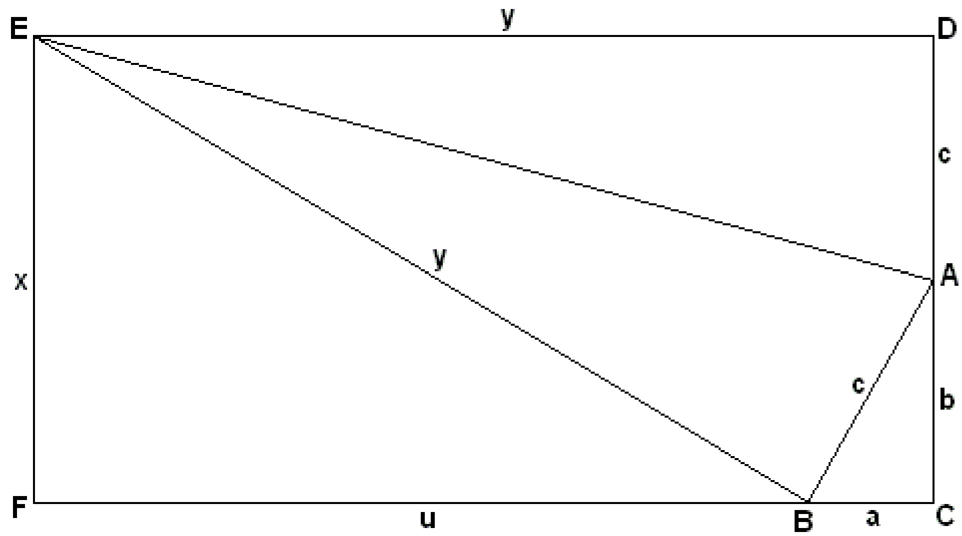
$$2a\cdot b/2 + 2b\cdot a/2 + 2\cdot c^2/2.$$

Equating the two gives $a^2 + b^2 = c^2$.

The proof is closely related to President Garfield's proof.

Proof #53

Larry Hoehn also published the following proof (*The Mathematics Teacher*, 88 (1995), p. 168.):



Extend the leg AC of the right triangle ABC to D so that AD = AB = c, as in the diagram. At D draw a perpendicular to CD. At A draw a bisector of the angle BAD. Let the two lines meet in E. Finally, let EF be perpendicular to CF.

By this construction, triangles ABE and ADE share side AE, have other two sides equal: AD = AB, as well as the angles formed by those sides: ∠BAE = ∠DAE. Therefore, triangles ABE and ADE are congruent by SAS. From here, angle ABE is right.

It then follows that in right triangles ABC and BEF angles ABC and EBF add up to 90°. Thus

$$\angle ABC = \angle BEF \text{ and } \angle BAC = \angle EBF.$$

The two triangles are similar, so that

$$x/a = u/b = y/c.$$

But, EF = CD, or $x = b + c$, which in combination with the above proportion gives

$$u = b(b + c)/a \text{ and } y = c(b + c)/a.$$

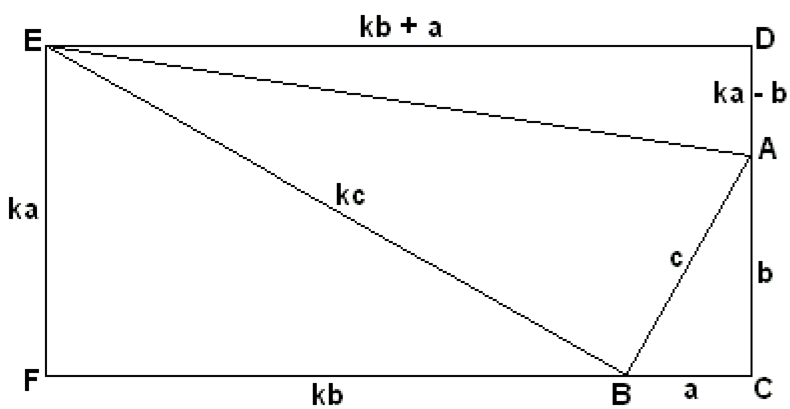
On the other hand, $y = u + a$, which leads to

$$c(b + c)/a = b(b + c)/a + a,$$

which is easily simplified to $c^2 = a^2 + b^2$.

Proof #54k

Later (*The Mathematics Teacher*, 90 (1997), pp. 438-441.) Larry Hoehn took a second look at his proof and produced a generic one, or rather a whole 1-parameter family of proofs, which, for various values of the parameter, included his older proof as well as #41. Below I offer a simplified variant inspired by Larry's work.



To reproduce the essential point of proof #53, i.e. having a right angled triangle ABE and another BEF, the latter being similar to ΔABC, we may simply place ΔBEF with sides ka, kb, kc, for some k, as shown in the diagram. For the diagram to make sense we should restrict k so that $ka \geq b$. (This insures that D does not go below A.)

Now, the area of the rectangle CDEF can be computed directly as the product of its sides ka and (kb + a), or as the sum of areas of triangles BEF, ABE, ABC, and ADE. Thus we get

$$ka\cdot(kb + a) = ka\cdot kb/2 + kc\cdot c/2 + ab/2 + (kb + a)\cdot(ka - b)/2,$$

which after simplification reduces to

$$a^2 = c^2/2 + a^2/2 - b^2/2,$$

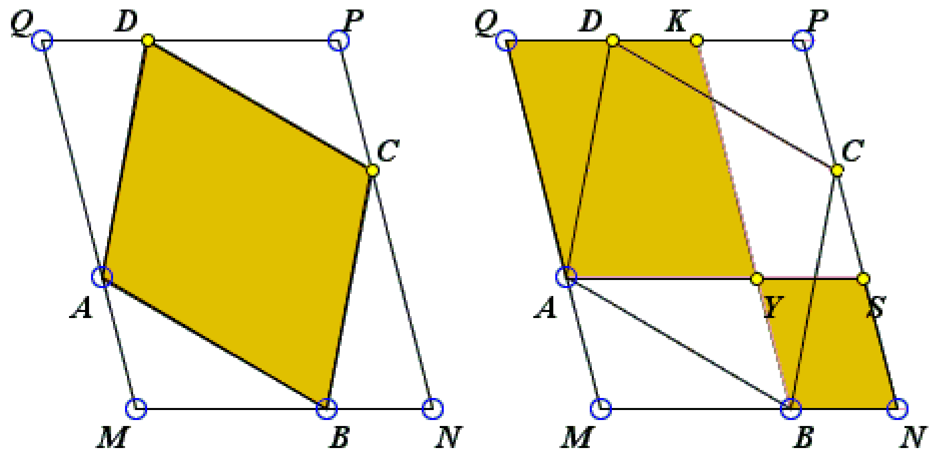
which is just one step short of the Pythagorean proposition.

The proof works for any value of k satisfying $k \ge b/a$. In particular, for $k = b/a$ we get proof #41. Further, $k = (b + c)/a$ leads to proof #53. Of course, we would get the same result by representing the area of the trapezoid AEFB in two ways. For $k = 1$, this would lead to President Garfield's proof.

Obviously, dealing with a trapezoid is less restrictive and works for any positive value of k.

Proof #55

The following generalization of the Pythagorean theorem is due to W. J. Hazard (*Am Math Monthly*, v 36, n 1, 1929, 32-34). The proof is a slight simplification of the published one.

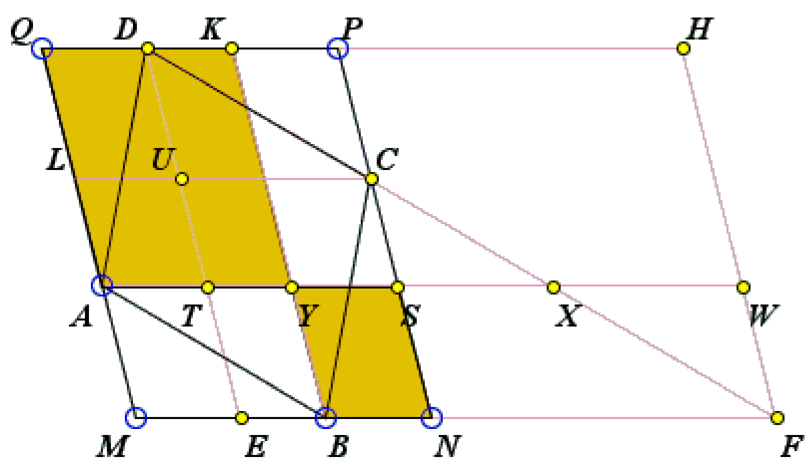


Let parallelogram ABCD inscribed into parallelogram MNPQ is shown on the left. Draw BK||MQ and AS||MN. Let the two intersect in Y. Then

Area(ABCD) = Area(QAYK) + Area(BNSY).

A reference to Proof #9 shows that this is a true generalization of the Pythagorean theorem. The diagram of Proof #9 is obtained when both parallelograms become squares.

The proof proceeds in 4 steps. First, extend the lines as shown below.

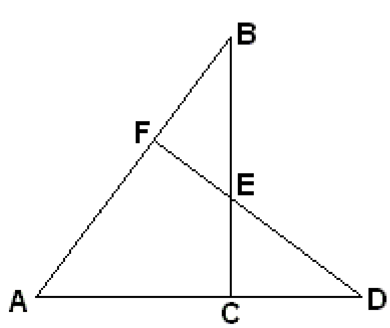


Then, the first step is to note that parallelograms ABCD and ABFX have equal bases and altitudes, hence equal areas (Euclid I.35 In fact, they are nicely equidecomposable.) For the same reason, parallelograms ABFX and YBFW also have equal areas. This is step 2. On step 3 observe that parallelograms SNFW and DTSP have equal areas. (This is because parallelograms DUCP and TENS are equal and points E, S, H are collinear. Euclid I.43 then implies equal areas of parallelograms SNFW and DTSP) Finally, parallelograms DTSP and QAYK are outright equal.

(There is a dynamic version of the proof.)

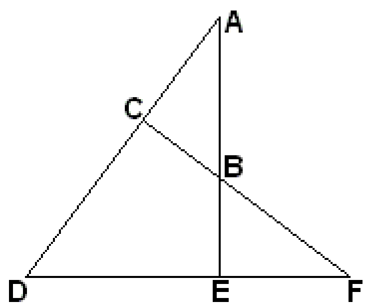
Proof #56

More than a hundred years ago *The American Mathematical Monthly* published a series of short notes listing great many proofs of the Pythagorean theorem. The authors, B. F. Yanney and J. A. Calderhead, went an extra mile counting and classifying proofs of various flavors. This and the next proof which are numbers V and VI from their collection (*Am Math Monthly*, v.3, n. 4 (1896), 110-113) give a sample of their thoroughness. Based on the diagram below they counted as many as 4864 different proofs. I placed a sample of their work on a separate page.



Proof #57

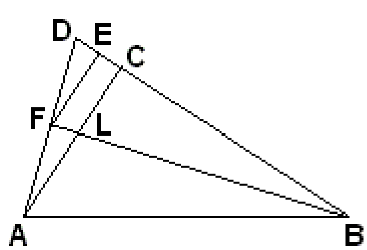
Treating the triangle a little differently, now extending its sides instead of crossing them, B. F. Yanney and J. A. Calderhead came up with essentially the same diagram:



Following the method they employed in the previous proof, they again counted 4864 distinct proofs of the Pythagorean proposition.

Proof #58

(B. F. Yanney and J. A. Calderhead, *Am Math Monthly*, v.3, n. 6/7 (1896), 169-171, #VII)



Let ABC be right angled at C. Produce BC making BD = AB. Join AD. From E, the midpoint of CD, draw a perpendicular meeting AD at F. Join BF. ΔADC is similar to ΔBFE. Hence.

$$AC/BE = CD/EF.$$

But CD = BD - BC = AB - BC. Using this

$BE = BC + CD/2$

$BE = BC + (AB - BC)/2$

$\quad = (AB + BC)/2$

and EF = AC/2. So that

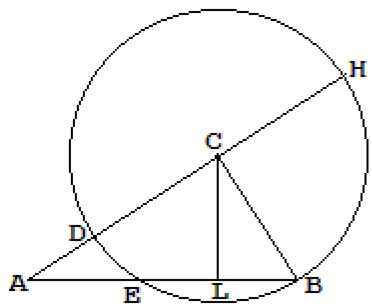
$$AC\cdot AC/2 = (AB - BC)\cdot(AB + BC)/2,$$

which of course leads to $AB^2 = AC^2 + BC^2$.

(As we've seen in proof 56, Yanney and Calderhead are fond of exploiting a configuration in as many ways as possible. Concerning the diagram of the present proof, they note that triangles BDF, BFE, and FDE are similar, which allows them to derive a multitude of proportions between various elements of the configuration. They refer to their approach in proof 56 to suggest that here too there are great many proofs based on the same diagram. They leave the actual counting to the reader.)

Proof #59

(B. F. Yanney and J. A. Calderhead, *Am Math Monthly*, v.3, n. 12 (1896), 299-300, #XVII)



Let ABC be right angled at C and let BC = a be the shortest of the two legs. With C as a center and a as a radius describe a circle. Let D be the intersection of AC with the circle, and H the other one obtained by producing AC beyond C, E the intersection of AB with the circle. Draw CL perpendicular to AB. L is the midpoint of BE.

By the Intersecting Chords theorem,

$$AH \cdot AD = AB \cdot AE.$$

In other words,

$$(b + a)(b - a) = c(c - 2 \cdot BL).$$

Now, the right triangles ABC and BCL share an angle at B and are, therefore, similar, wherefrom

$$BL/BC = BC/AB,$$

so that $BL = a^2/c$. Combining all together we see that

$$b^2 - a^2 = c(c - 2a^2/c)$$

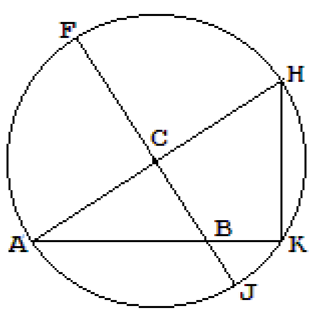
and ultimately the Pythagorean identity.

Remark

Note that the proof fails for an isosceles right triangle. To accommodate this case, the authors suggest to make use of the usual method of the theory of limits. I am not at all certain what is the "usual method" that the authors had in mind. Perhaps, it is best to subject this case to Socratic reasoning which is simple and does not require the theory of limits. If the case is exceptional anyway, why not to treat it as such.

Proof #60

(B. F. Yanney and J. A. Calderhead, *Am Math Monthly*, v.3, n. 12 (1896), 299-300, #XVIII)



The idea is the same as before (proof #59), but now the circle has the radius b, the length of the longer leg. Having the sides produced as in the diagram, we get

$$AB\cdot BK = BJ\cdot BF,$$

or

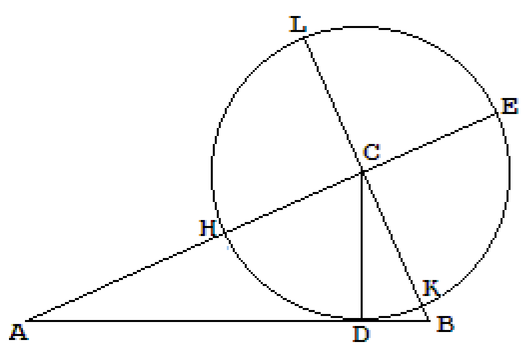
$c \cdot BK = (b - a)(b + a).$

BK, which is AK - c, can be found from the similarity of triangles ABC and AKH: $AK = 2b^2/c$.

Note that, similar to the previous proof, this one, too, dos not work in case of the isosceles triangle.

Proof #61

(B. F. Yanney and J. A. Calderhead, *Am Math Monthly*, v.3, n. 12 (1896), 299-300, #XIX)



This is a third in the family of proofs that invoke the Intersecting Chords theorem. The radius of the circle equals now the altitude from the right angle C. Unlike in the other two proofs, there are now no exceptional cases. Referring to the diagram,

$$AD^2 = AH \cdot AE = b^2 - CD^2,$$

$$BD^2 = BK \cdot BL = a^2 - CD^2,$$

$$2AD \cdot BD = 2CD^2.$$

Adding the three yields the Pythagorean identity.

Proof #62

This proof, which is due to Floor van Lamoen, makes use of some of the many properties of the symmedian point. First of all, it is known that in any triangle ABC the symmedian point K has the barycentric coordinates proportional to the squares of the triangle's side lengths. This implies a relationship between the areas of triangles ABK, BCK and ACK:

$$\text{Area}(BCK) : \text{Area}(ACK) : \text{Area}(ABK) = a^2 : b^2 : c^2.$$

Next, in a right triangle, the symmedian point is the midpoint of the altitude to the hypotenuse. If, therefore, the angle at C is right and CH is the altitude (and also the symmedian) in question, AK serves as a median of ΔACH and BK as a median of ΔBCH. Recollect now that a median cuts a triangle into two of equal areas. Thus,

$$\text{Area}(ACK) = \text{Area}(AKH) \text{ and}$$
$$\text{Area}(BCK) = \text{Area}(BKH).$$

But

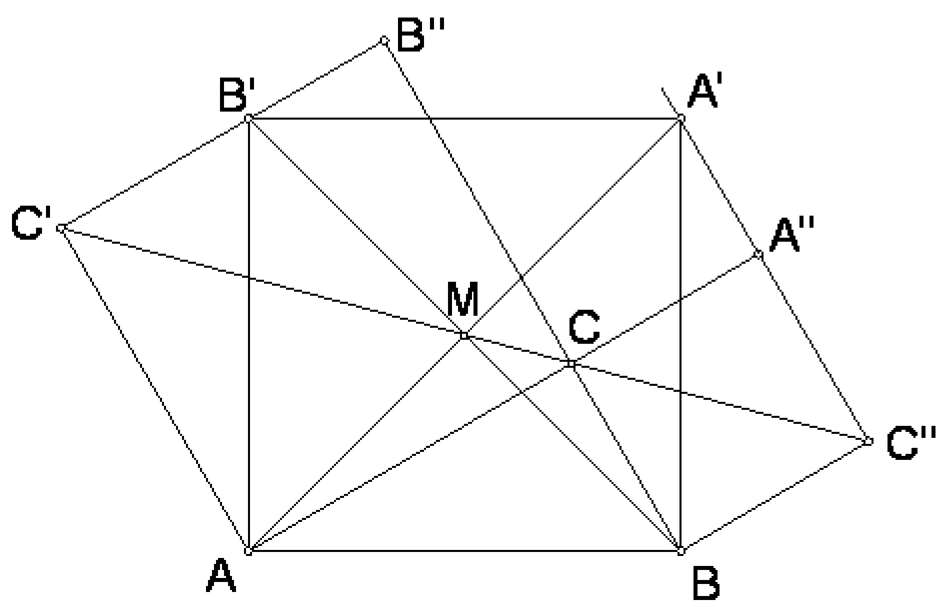
$$\begin{aligned}\text{Area}(ABK) &= \text{Area}(AKH) + \text{Area}(BKH)\\ &= \text{Area}(ACK) + \text{Area}(BCK),\end{aligned}$$

so that indeed $k\cdot c^2 = k\cdot a^2 + k\cdot b^2$, for some $k > 0$; and the Pythagorean identity follows.

Floor also suggested a different approach to exploiting the properties of the symmedian point. Note that the symmedian point is the center of gravity of three weights on A, B and C of magnitudes a^2, b^2 and c^2 respectively. In the right triangle, the foot of the altitude from C is the center of gravity of the weights on B and C. The fact that the symmedian point is the midpoint of this altitude now shows that $a^2 + b^2 = c^2$.

Proof #63

This is another proof by Floor van Lamoen; Floor has been led to the proof via Bottema's theorem. However, the theorem is not actually needed to carry out the proof.



In the figure, M is the center of square ABA'B'. Triangle AB'C' is a rotation of triangle ABC. So we see that B' lies on C'B''. Similarly, A' lies on A''C''. Both AA'' and BB'' equal a + b. Thus the distance from M to AC' as well as to B'C' is equal to (a + b)/2. This gives

Area(AMB'C') = Area(MAC') + Area(MB'C')

= (a + b)/2 · b/2 + (a + b)/2 · a/2

= $a^2/4 + ab/2 + b^2/4$.

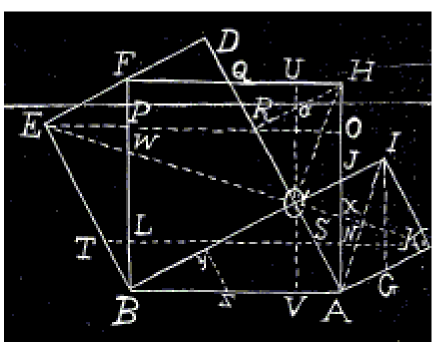
But also:

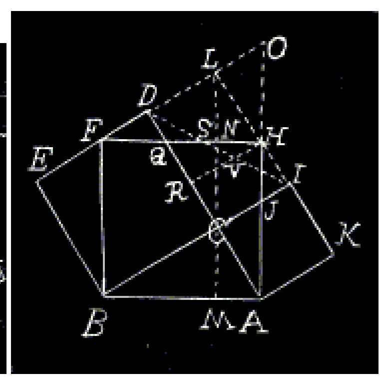
Area(AMB'C') = Area(AMB') + Area(AB'C')

= $c^2/4 + ab/2$.

This yields $a^2/4 + b^2/4 = c^2/4$ and the Pythagorean theorem.

The basic configuration has been exploited by B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v.4, n 10, (1987), 250-251) to produce several proofs based on the following diagrams

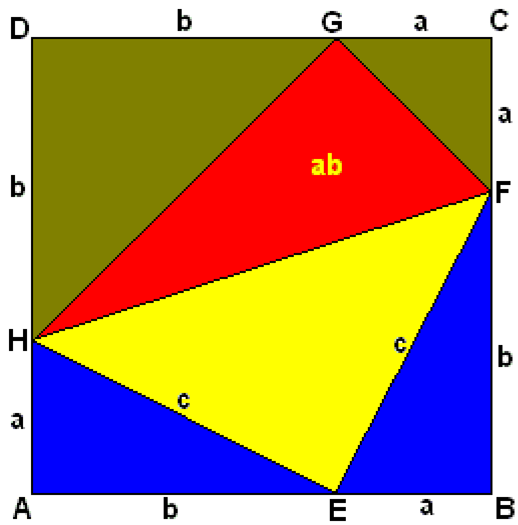




None of their proofs made use of the centrality of point M.

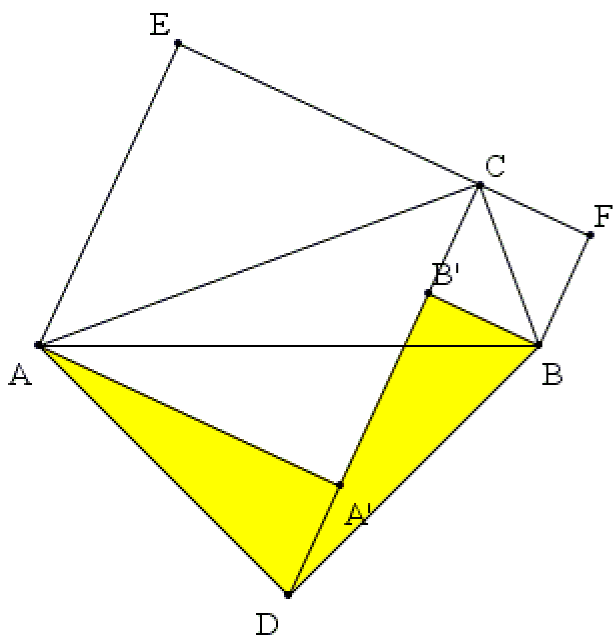
Proof #64

And yet one more proof by Floor van Lamoen; in a quintessentially mathematical spirit, this time around Floor reduces the general statement to a particular case, that of a right isosceles triangle. The latter has been treated by Socrates and is shown independently of the general theorem.



FH divides the square ABCD of side a + b into two equal quadrilaterals, ABFH and CDHF. The former consists of two equal triangles with area ab/2, and an isosceles right triangle with area $c^2/2$. The latter is composed of two isosceles right triangles: one of area $a^2/2$, the other $b^2/2$, and a right triangle whose area (by the introductory remark) equals ab! Removing equal areas from the two quadrilaterals, we are left with the identity of areas: $a^2/2 + b^2/2 = c^2/2$.

The idea of Socrates' proof that the area of an isosceles right triangle with hypotenuse k equals $k^2/4$, has been used before, albeit implicitly. For example, Loomis, #67 (with a reference to the 1778 edition of E. Fourrey's *Curiosities Geometrique* [Loomis' spelling]) relies on the following diagram:



Triangle ABC is right at C, while ABD is right isosceles. (Point D is the midpoint of the semicircle with diameter AB, so that CD is the bisector of the right angle ACB.) AA' and BB' are perpendicular to CD, and AA'CE and BB'CF are squares; in particular EF ⊥ CD.

Triangles AA'D and DB'B (having equal hypotenuses and complementary angles at D) are congruent. It follows that AA' = B'D = A'C = CE = AE. And similar for the segments equal to B'C. Further, CD = B'C + B'D = CF + CE = EF.

Area(ADBC) = Area(ADC) + Area(DBC)

Area(ADBC) = CD×AA'/2 + CD×BB'/2

Area(ADBC) = CD×EF/2.

On the other hand,

Area(ABFE) = EF×(AE + BF)/2

Area(ADBC) = CD×AA'/2 + CD×BB'/2

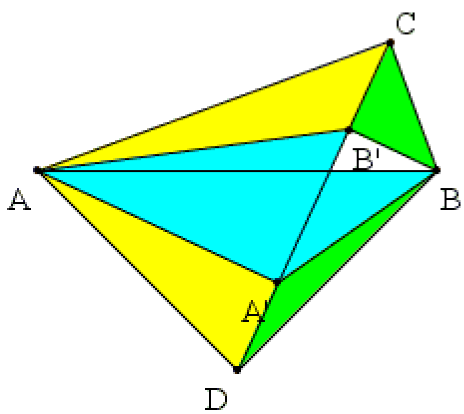
Area(ADBC) = CD×EF/2.

Thus the two quadrilateral have the same area and ΔABC as the intersection. Removing ΔABC we see that

Area(ADB) = Area(ACE) + Area(BCF).

The proof reduces to Socrates' case, as the latter identity is equivalent to $c^2/4 = a^2/4 + b^2/4$.

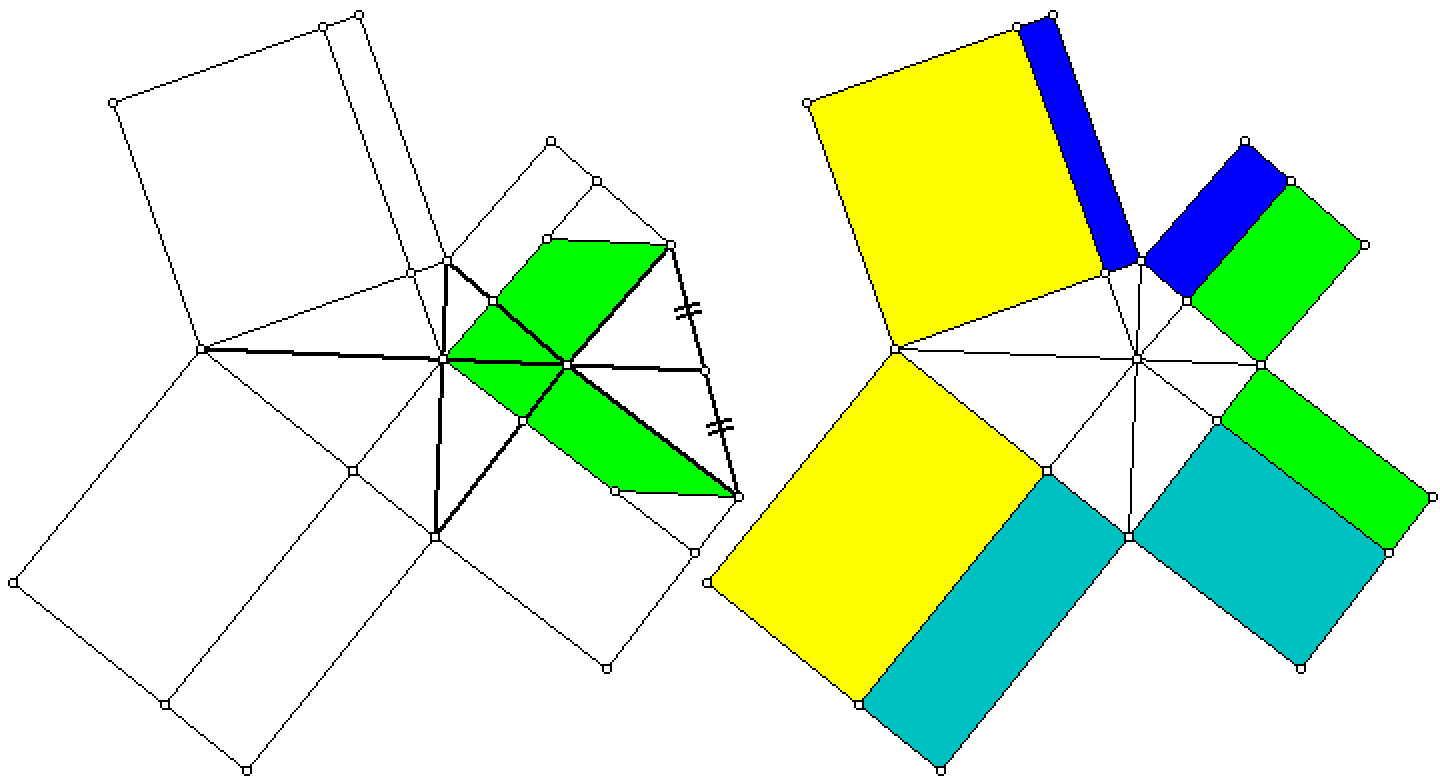
More recently, Bui Quang Tuan came up with a different argument:



From the above, Area(BA'D) = Area(BB'C) and Area(AA'D) = Area(AB'C). Also, Area(AA'B) = Area(AA'B'), for AA'||BB'. It thus follows that Area(ABD) = Area(AA'C) + Area(BB'C), with the same consequences.

Proof #65

This and the following proof are also due to Floor van Lamoen. Both a based on the following lemma, which appears to generalize the Pythagorean theorem: Form squares on the sides of the *orthodiagonal quadrilateral*. The squares fall into two pairs of opposite squares. Then the sum of the areas of the squares in two pairs are equal.

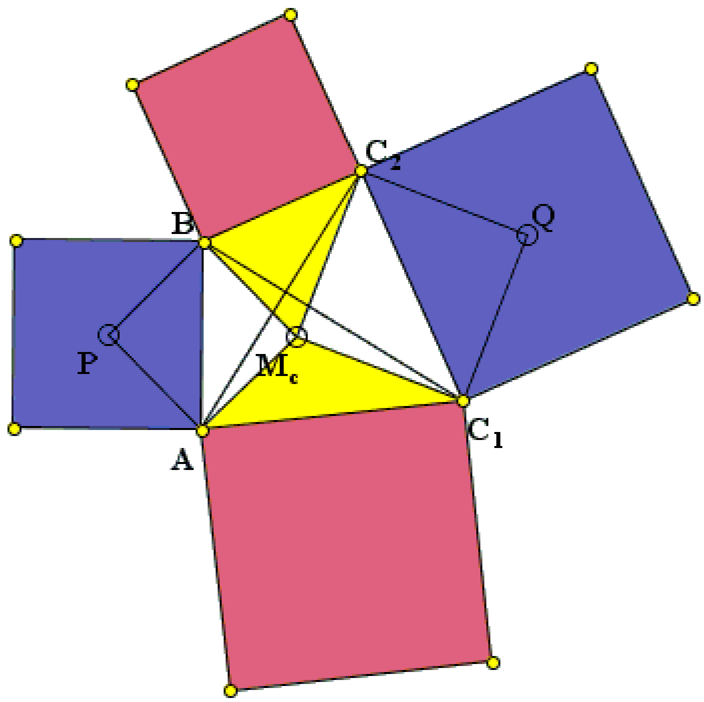


The proof is based on the friendly relationship between a triangle and its flank triangles: the altitude of a triangle through the right angle extended beyond the vertex is the median of the flank triangle at the right angle. With this in mind, note that the two parallelograms in the left figure not only share the base but also have equal altitudes. Therefore they have equal areas. Using shearing, we see that the squares at hand split into pairs of rectangles of equal areas, which can be combined in two ways proving the lemma.

For the proof now imagine two adjacent vertices of the quadrilateral closing in towards the point of intersection of the diagonals. In the limit, the quadrilateral will become a right triangle and one of the squares shrink to a point. Of the remaining three squares two will add up to the third.

Proof #66

(Floor van Lamoen). The lemma from Proof 65 can be used in a different way:



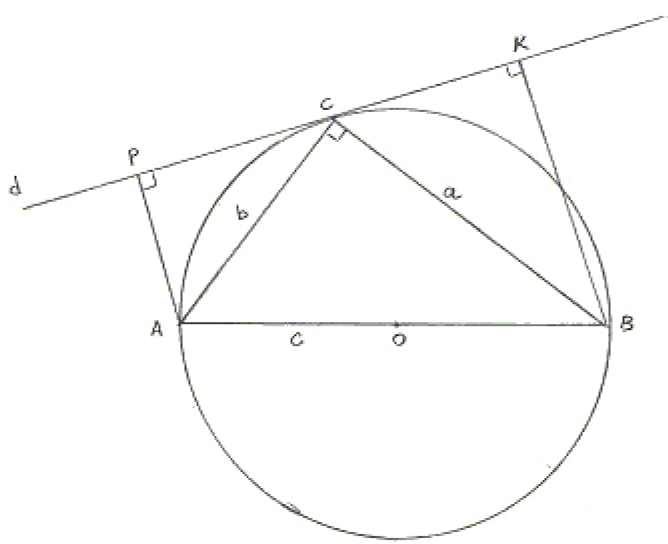
Let there be two squares: $APBM_c$ and $C_1M_cC_2Q$ with a common vertex M_c. Rotation through 90° in the positive direction around M_c moves C_1M_c into C_2M_c and BM_c into AM_c. This implies that ΔBM_cC_1 rotates into ΔAM_cC_2 so that AC_2 and BC_1 are orthogonal. Quadrilateral ABC_2C_1 is thus orthodiagonal and the lemma applies: the red and blue squares add up to the same area. The important point to note is that the sum of the areas of the original squares $APBM_c$ and $C_1M_cC_2Q$ is half this quantity.

Now assume the configurations is such that M_c coincides with the point of intersection of the diagonals. Because of the resulting symmetry, the red squares are equal. Therefore, the areas of $APBM_c$ and $C_1M_cC_2Q$ add up to that of a red square!

(There is a dynamic illustration of this argument.)

Proof #67

This proof was sent to me by a 14 year old Sina Shiehyan from Sabzevar, Iran. The circumcircle aside, the combination of triangles is exactly the same as in S. Brodie's subcase of Euclid's VI.31. However, Brodie's approach if made explicit would require argument different from the one employed by Sina. So, I believe that her derivation well qualifies as an individual proof.



From the endpoints of the hypotenuse AB drop perpendiculars AP and BK to the tangent to the circumcircle of ΔABC at point C. Since OC is also perpendicular to the tangent, C is the midpoint of KP. It follows that

$$\begin{aligned}\text{Area}(ACP) + \text{Area}(BCK) &= CP\cdot AP/2 + CK\cdot BK/2\\ &= [KP\cdot(AP + BK)/2]/2\\ &= \text{Area}(ABKP)/2.\end{aligned}$$

Therefore, Area(ABC) is also Area(ABKP)/2. So that

$$\text{Area}(ACP) + \text{Area}(BCK) = \text{Area}(ABC)$$

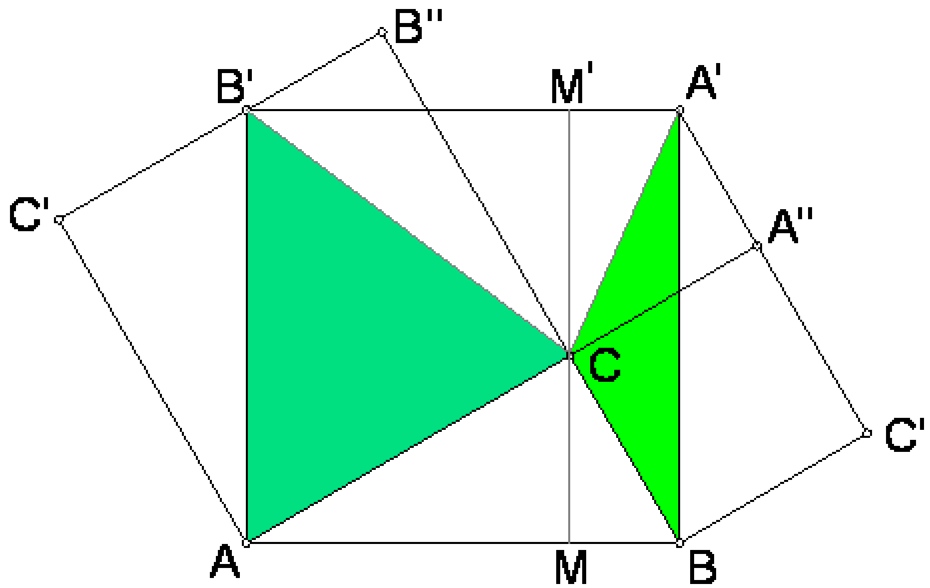
Now all three triangles are similar (as being right and having equal angles), their areas therefore related as the squares of their hypotenuses, which are b, a, and c respectively. And the theorem follows.

I have placed the original Sina's derivation on a separate page.

Proof #68

The Pythagorean theorem is a direct consequence of the Parallelogram Law. I am grateful to Floor van Lamoen for bringing to my attention a proof without words for the latter. There is a second proof which I love even better.

Proof #69

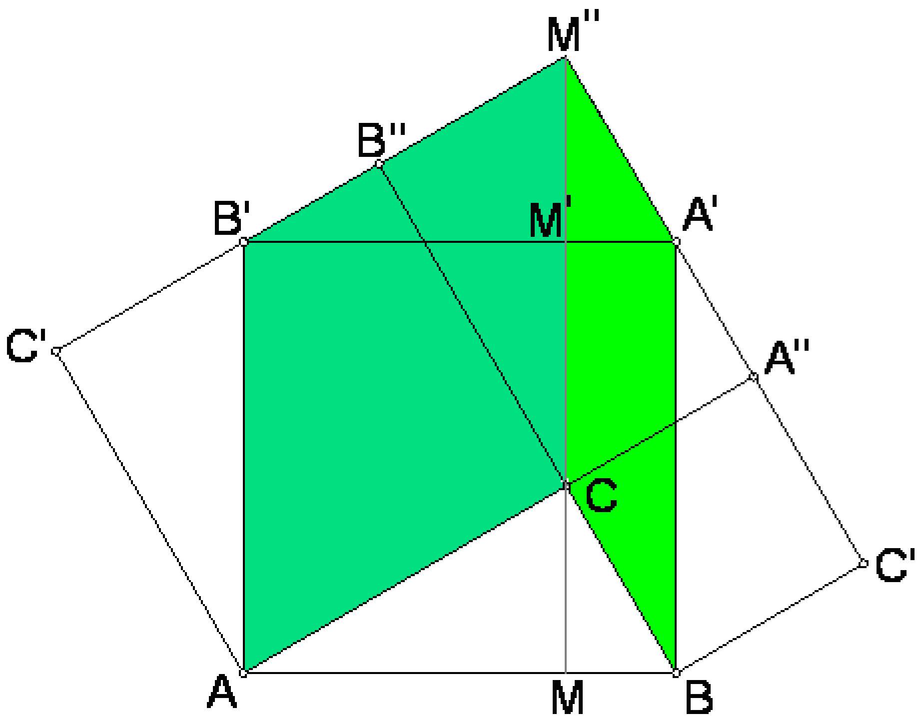


Twice in his proof of I.47 Euclid used the fact that if a parallelogram and a triangle share the same base and are in the same parallels (I.41), the area of the parallelogram is twice that of the triangle. Wondering at the complexity of the setup that Euclid used to employ that argument, Douglas Rogers came up with a significant simplification that Euclid without a doubt would prefer if he saw it.

Let ABA'B', ACB"C', and BCA"C" be the squares constructed on the hypotenuse and the legs of ΔABC as in the diagram below. As we saw in proof 63, B' lies on C'B" and A' on A"C". Consider triangles BCA' and ACB'. On one hand, one shares the base BC and is in the same parallels as the parallelogram (a square actually) BCA"C". The other shares the base AC and is in the same parallels as the parallelogram ACB"C'. It thus follows by Euclid's argument that the total area of the two triangles equals half the sum of the areas of the two squares. Note that the squares are those constructed on the legs of ΔABC.

On the other hand, let MM' pass through C parallel to AB' and A'B. Then the same triangles BCA' and ACB' share the base and are in the same parallels as parallelograms (actually rectangles) MBA'M'and AMM'B', respectively. Again employing Euclid's argument, the area of the triangles is half that of the rectangles, or half that of the square ABA'B'. And we are done.

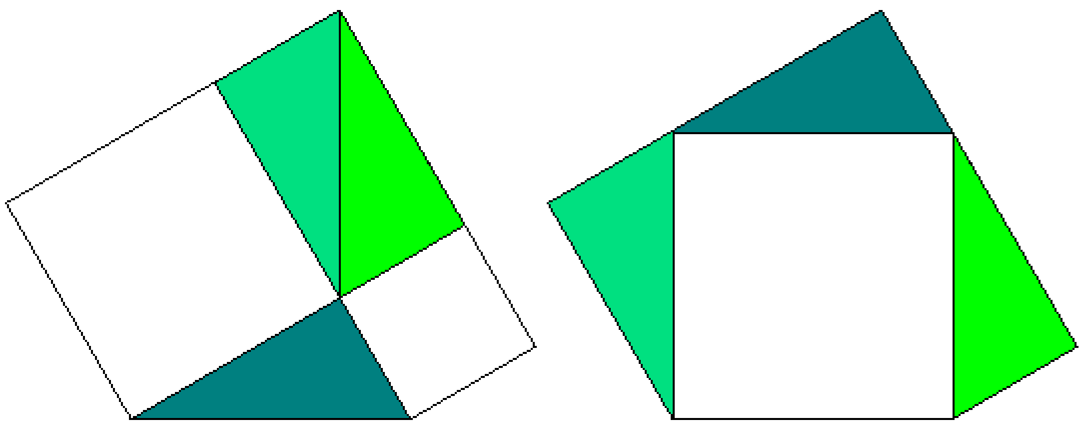
As a matter of fact, this is one of the family of 8 proofs inserted by J. Casey in his edition of Euclid's *Elements*. I placed the details on a separate page.



Now, it appears that the argument can be simplified even further by appealing to the more basic (I.35): *Parallelograms which are on the same base and in the same parallels equal one another*. The side lines C'B" and A'C" meet at point M" that lies on MM', see, e.g. proof 12 and proof 24. Then by (I.35) parallelograms AMM'B', ACM"B' and ACB"C' have equal areas and so do parallelograms MBA'M', BA'M"C, and BC"A"C. Just what is needed.

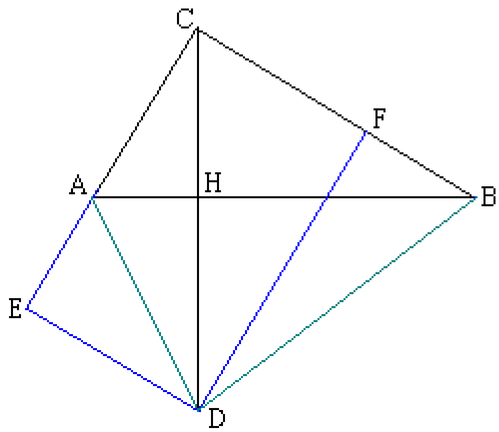
The latter approach reminds one of proof 37, but does not require any rotation and does the shearing "in place". The dynamic version and the unfolded variant of this proof appear on separate pages.

In a private correspondence, Kevin "Starfox" Arima pointed out that sliding triangles is a more intuitive operation than shearing. Moreover, a proof based on a rearrangement of pieces can be performed with paper and scissors, while those that require shearing are confined to drawings or depend on programming, e.g. in Java. His argument can be represented by the following variant of both this proof and # 24.



A dynamic illustration is also available.

Proof #70



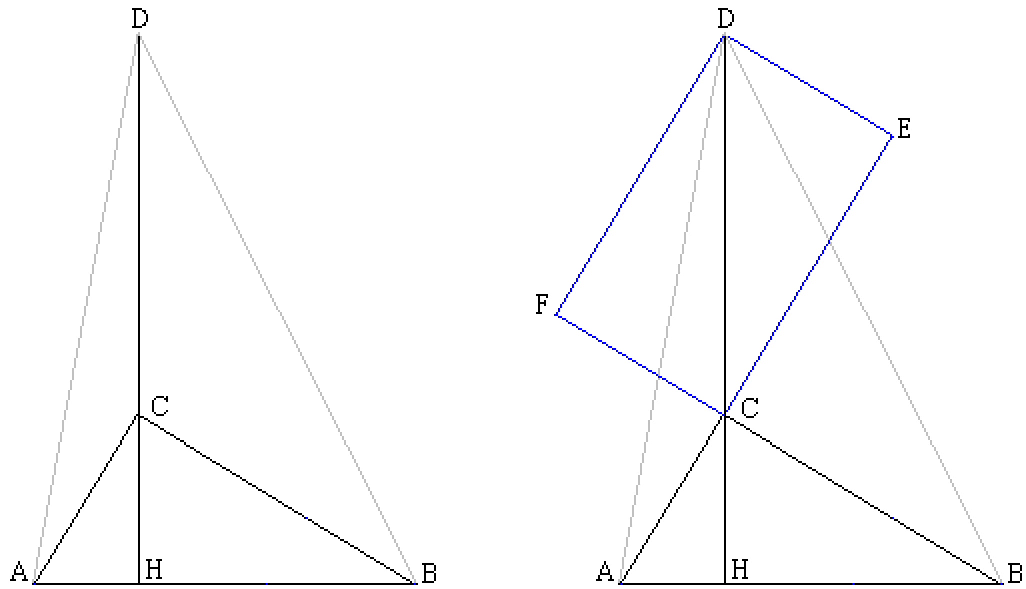
Extend the altitude CH to the hypotenuse to D: CD = AB and consider the area of the orthodiagonal quadrilateral ACBD (similar to proofs 47-49.) On one hand, its area equals half the product of its diagonals: $c^2/2$. On the other, it's the sum of areas of two triangles, ACD and BCD. Drop the perpendiculars DE and DF to AC and BC. Rectangle CEDF is has sides equal DE and DF equal to AC and BC, respectively, because for example ΔCDE = ΔABC as both are right, have equal hypotenuse and angles. It follows that

Area(CDA) = b^2 and

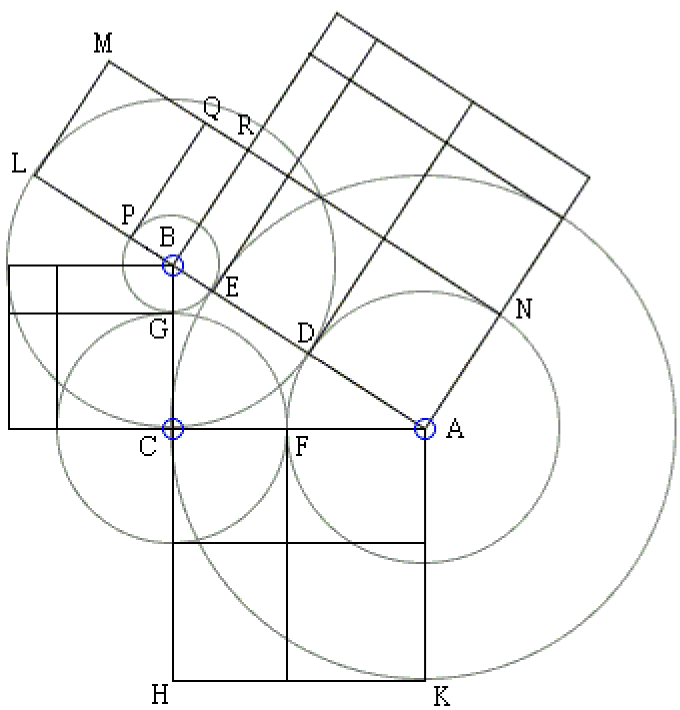
Area(CDB) = a^2

so that indeed $c^2/2 = a^2/2 + b^2/2$.

This is proof 20 from Loomis' collection. In proof 29, CH is extended upwards to D so that again CD = AB. Again the area of quadrilateral ACBD is evaluated in two ways in exactly same manner.



Proof #71



Let D and E be points on the hypotenuse AB such that BD = BC and AE = AC. Let AD = x, DE = y, BE = z. Then AC = $x + y$, BC = $y + z$, AB = $x + y + z$. The Pythagorean theorem is then equivalent to the algebraic identity

$$(y + z)^2 + (x + y)^2 = (x + y + z)^2.$$

Which simplifies to

$$y^2 = 2xz.$$

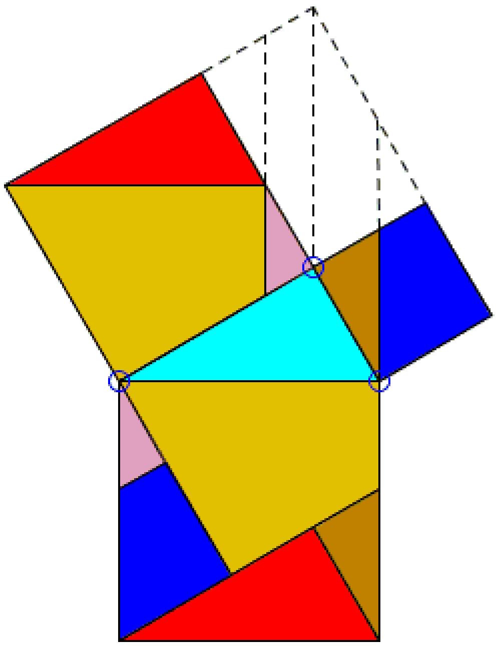
To see that the latter is true calculate the power of point A with respect to circle B(C), i.e. the circle centered at B and passing through C, in two ways: first, as the square of the tangent AC and then as the product AD·AL:

$$(x + y)^2 = x(x + 2(y + z)),$$

which also simplifies to $y^2 = 2xz$.

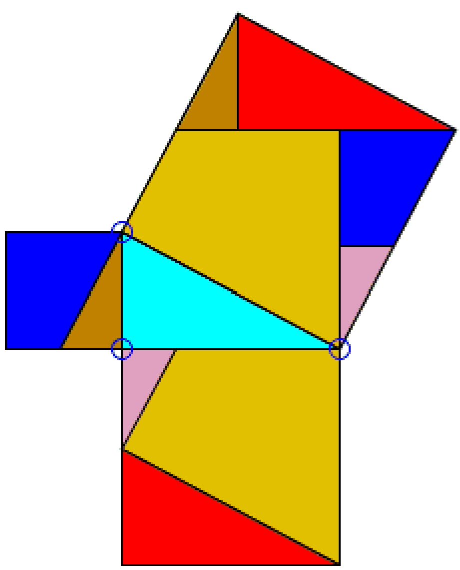
This is algebraic proof 101 from Loomis' collection. Its dynamic version is available separately.

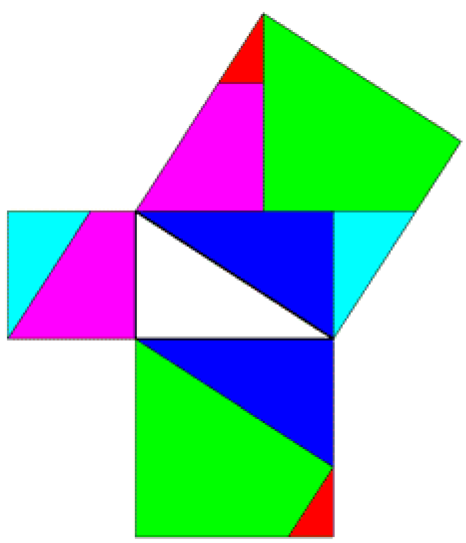
Proof #72



This is geometric proof #25 from E. S. Loomis' collection, for which he credits an earlier publication by J. Versluys (1914). The proof is virtually self-explanatory and the addition of a few lines shows a way of making it formal.

Michel Lasvergnas came up with an even more ransparent rearrangement (on the right below):

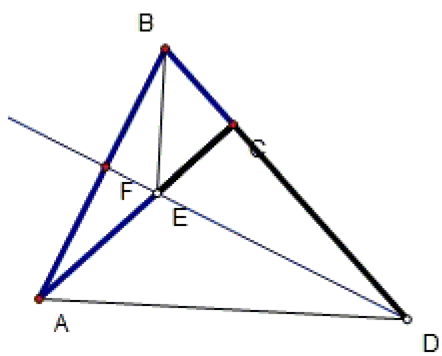




These two are obtained from each other by rotating each of the squares 180° around its center.

A dynamic version is also available.

Proof #73



This proof is by weininjieda from Yingkou, China who plans to become a teacher of mathematics, Chinese and history. It was included as algebraic proof #50 in E. S. Loomis' collection, for which he refers to an earlier publication by J. Versluys (1914), where the proof is credited to Cecil Hawkins (1909) of England.

Let CE = BC = a, CD = AC = b, F is the intersection of DE and AB.

ΔCED = ΔABC, hence DE = AB = c. Since, AC ⊥ BD and BE ⊥AD, ED ⊥AB, as the third altitude in ΔABD. Now from

$$\text{Area}(\Delta ABD) = \text{Area}(\Delta ABE) + \text{Area}(\Delta ACD) + \text{Area}(\Delta BCE)$$

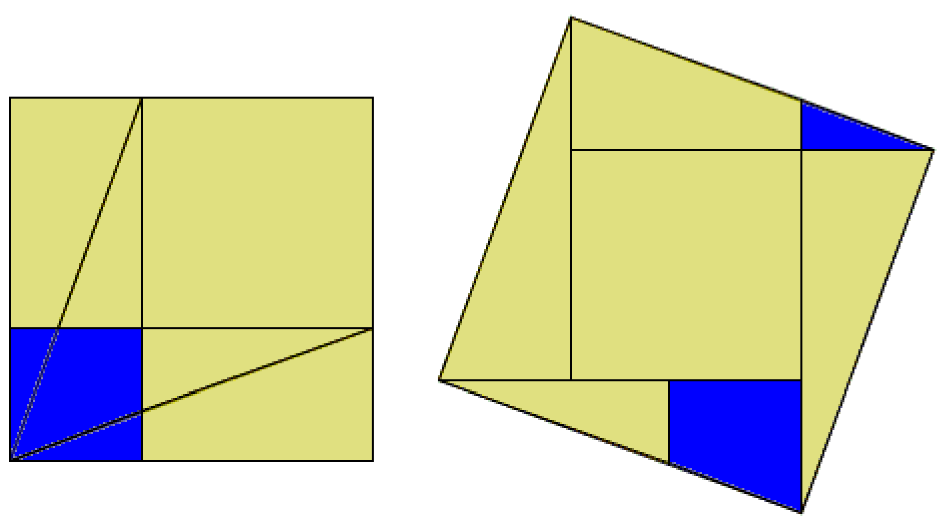
we obtain

$$c(c + EF) = EF \cdot c + b^2 + a^2,$$

which implies the Pythagorean identity.

Proof #74

The following proof by dissection is due to the 10^{th} century Persian mathematician and astronomer Abul Wafa (Abu'l-Wafa and also Abu al-Wafa) al-Buzjani. Two equal squares are easily combined into a bigger square in a way known yet to Socrates. Abul Wafa method works if the squares are different. The squares are placed to share a corner and two sidelines. They are cut and reassembled as shown. The dissection of the big square is almost the same as by Liu Hui. However, the smaller square is cut entirely differently. The decomposition of the resulting square is practically the same as that in Proof #3.

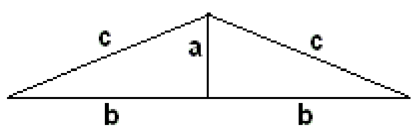


A dynamic version is also available.

Proof #75

This an additional application of Heron's formula to proving the Pythagorean theorem. Although it is much shorter than the first one, I placed it too in a separate file to facilitate the comparison.

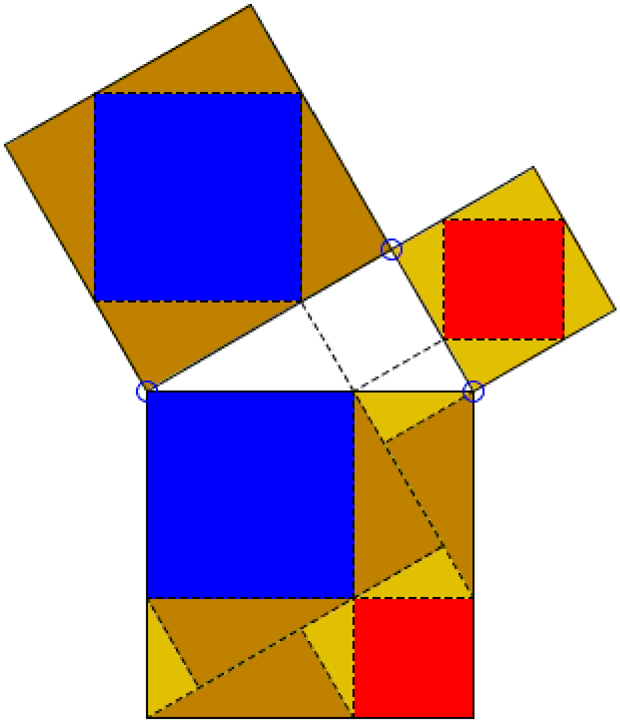
The idea is simple enough: Heron's formula applies to the isosceles triangle depicted in the diagram below.



Proof #76

This is a geometric proof #27 from E. S. Loomis' collection. According to Loomis, he received the proof in 1933 from J. Adams, The Hague. Loomis makes a remark pointing to the uniqueness of this proof among other dissections in that all the lines are either parallel or perpendicular to the sides of the given triangle. Which is strange as, say, proof #72 accomplishes they same feat and with fewer lines at that. Even more surprisingly the latter is also included into E. S. Loomis' collection as the geometric proof #25.

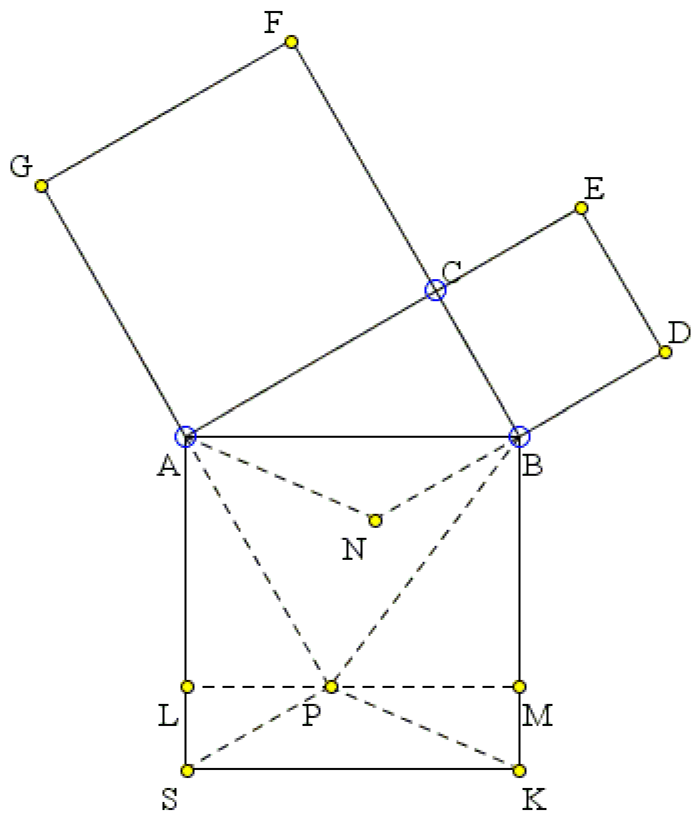
Inexplicably Loomis makes a faulty introduction to the construction starting with the wrong division of the hypotenuse. However, it is not difficult to surmise that the point that makes the construction work is the foot of the right angle bisector.

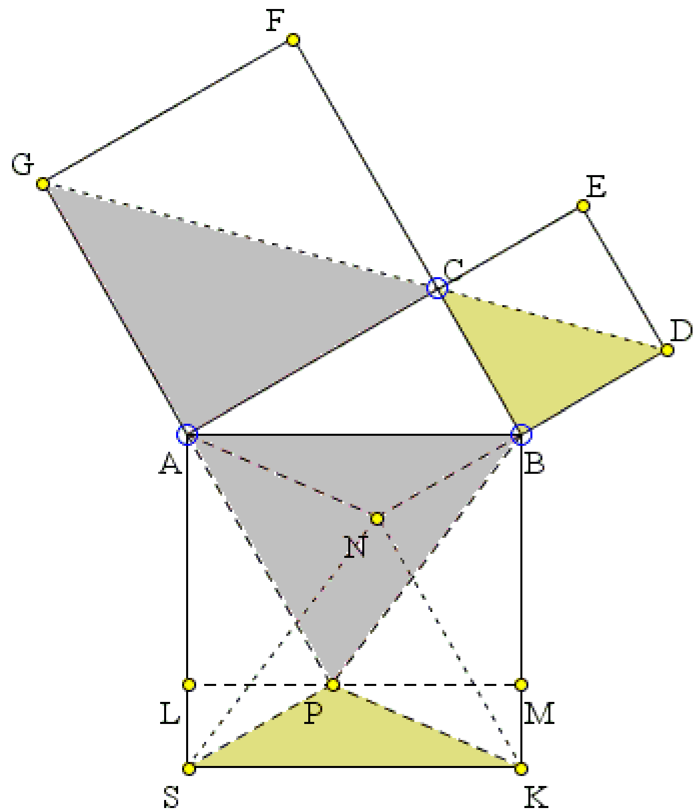


A dynamic illustration is available on a separate page.

Proof #77

This proof is by the famous Dutch mathematician, astronomer and physicist Christiaan Huygens (1629 � 1695) published in 1657. It was included in Loomis' collection as geometric proof #31. As in Proof #69, the main instrument in the proof is Euclid's I.41: if a parallelogram and a triangle that share the same base and are in the same parallels (I.41), the area of the parallelogram is twice that of the triangle.





More specifically,

Area(ABML) = 2·Area(ΔABP) = Area(ACFG), and

Area(KMLS) = 2·Area(ΔKPS), while

Area(BCED) = 2·Area(ΔANB).

Combining these with the fact that ΔKPS = ΔANB, we immediately get the Pythagorean proposition.

(A dynamic illustration is available on a separate page.)

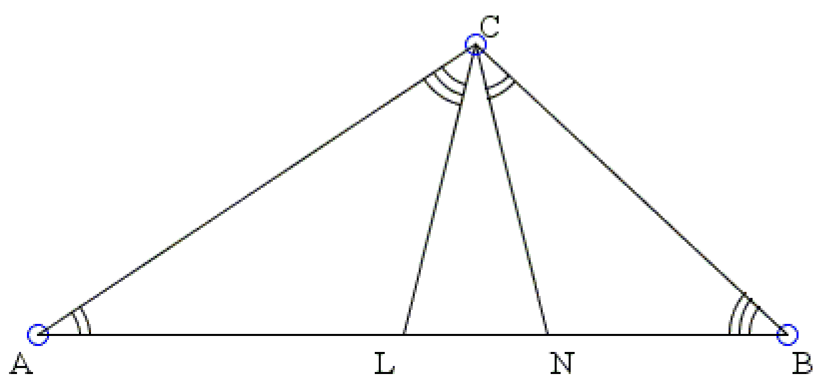
Proof #78

This proof is by the distinguished Dutch mathematician E. W. Dijkstra (1930 � 2002). The proof itself is, like Proof #18, a generalization of Proof #6 and is based on the same diagram. Both proofs reduce to a variant of Euclid VI.31 for right triangles (with the right angle at C). The proof aside, Dijkstra also found a remarkably fresh viewpoint on the essence of the theorem itself:

If, in a triangle, angles α, β, γ lie opposite the sides of length a, b, c, then

$$\text{sign}(\alpha + \beta - \gamma) = \text{sign}(a^2 + b^2 - c^2),$$

where sign(t) is the signum function.



As in Proof #18, Dijkstra forms two triangles ACL and BCN similar to the base ΔABC:

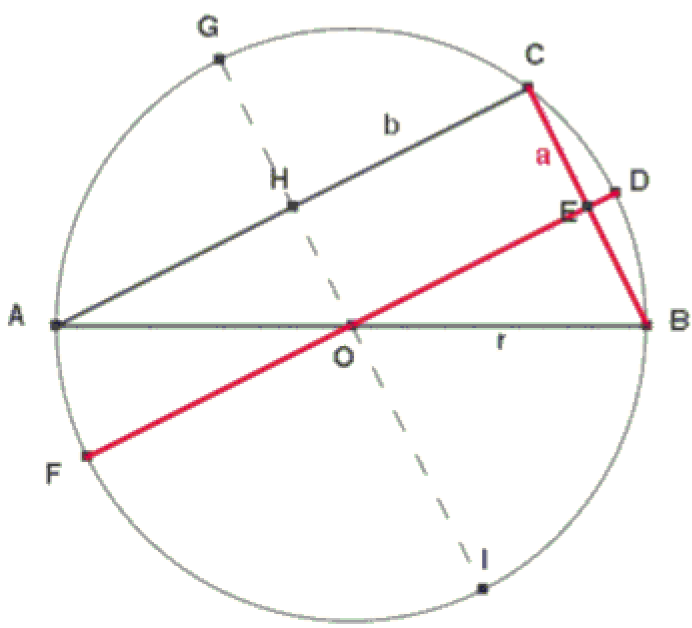
$\measuredangle BCN = \measuredangle CAB$ and

$\measuredangle ACL = \measuredangle CBA$

so that $\measuredangle ACB = \measuredangle ALC = \measuredangle BNC$. The details and a dynamic illustration are found in a separate page.

Proof #79

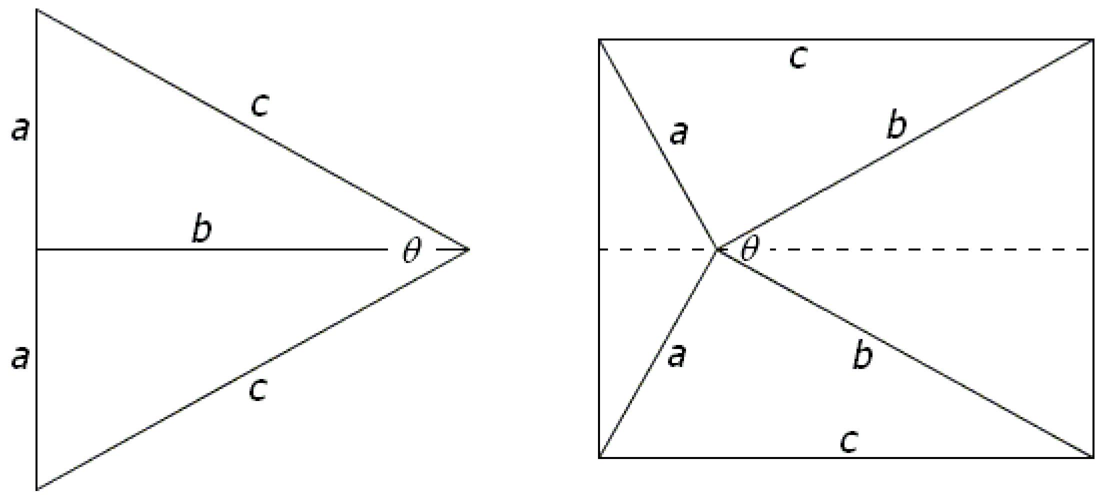
There are several proofs on this page that make use of the Intersecting Chords theorem, notably proofs ##59, 60, and 61, where the circle to whose chords the theorem applied had the radius equal to the short leg of ΔABC, the long leg and the altitude from the right angle, respectively. Loomis' book lists these among its collection of algebraic proofs along with several others that derive the Pythagorean theorem by means of the Intersecting Chords theorem applied to chords in a fanciful variety of circles added to ΔABC. Alexandre Wajnberg from Unité de Recherches sur l'Enseignement des Mathématiques, Université Libre de Bruxelles came up with a variant that appears to fill an omission in this series of proofs. The construction also looks simpler and more natural than any listed by Loomis. What a surprise!



For the details, see a separate page.

Proof #80

A proof based on the diagram below has been published in a letter to *Mathematics Teacher* (v. 87, n. 1, January 1994) by J. Grossman. The proof has been discovered by a pupil of his David Houston, an eighth grader at the time.



I am grateful to Professor Grossman for bringing the proof to my attention. The proof and a discussion appear in a separate page, but its essence is as follows.

Assume two copies of the right triangle with legs a and b and hypotenuse c are placed back to back as shown in the left diagram. The isosceles triangle so formed has the area $S = c^2 \sin(\theta) / 2$. In the right diagram, two copies of the same triangle are joined at the right angle and embedded into a rectangle with one side equal c. Each of the triangles has the area equal to half the area of half the rectangle, implying that the sum of the areas of the remaining isosceles triangles also add up to half the area of the rectangle, i.e., the area of the isosceles triangle in the left diagram. The sum of the areas of the two smaller isosceles triangles equals

$$S = a^2 \sin(\pi - \theta) / 2 + b^2 \sin(\theta) / 2$$

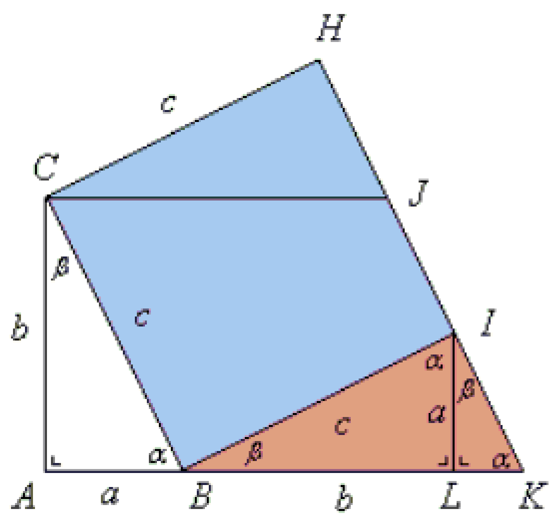
$$= (a^2 + b^2) \sin(\theta) / 2,$$

for, $\sin(\pi - \theta) = \sin(\theta)$. Since the two areas are equal and $\sin(\theta) \neq 0$, for a non-degenerate triangle, $a^2 + b^2 = c^2$.

Is this a *trigonometric proof*?

Proof #81

Philip Voets, an 18 years old law student from Holland sent me a proof he found a few years earlier. The proof is a combination of shearing employed in a number of other proofs and the decomposition of a right triangle by the altitude from the right angle into two similar pieces also used several times before. However, the accompanying diagram does not appear among the many in Loomis' book.



Given ΔABC with the right angle at A, construct a square BCHI and shear it into the parallelogram BCJK, with K on the extension of AB. Add IL perpendicular to AK. By the construction,

$$\text{Area}(BCJK) = \text{Area}(BCHI) = c^2.$$

On the other hand, the area of the parallelogram BCJK equals the product of the base BK and the altitude CA. In the right triangles BIK and BIL, BI = BC = c and ∠IBL = ∠ACB = β, making the two respectively similar and equal to ΔABC. ΔIKL is then also similar to ΔABC, and we find BL = b and LK = a^2/b. So that

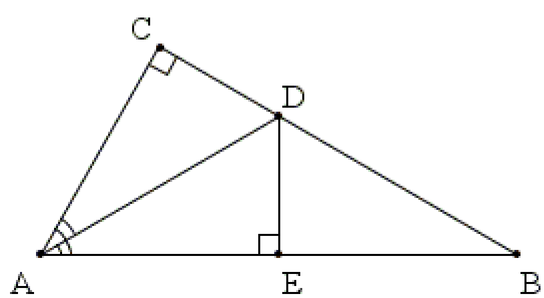
$$\begin{aligned}\text{Area}(BCJK) &= BK \times CA \\ &= (b + a^2/b) \times b \\ &= b^2 + a^2.\end{aligned}$$

We see that c^2 = Area(BCJK) = $a^2 + b^2$ completing the proof.

Proof #82

This proof has been published in the *American Mathematical Monthly* (v. 116, n. 8, 2009, October 2009, p. 687), with an Editor's note: Although this proof does not appear to be widely known, it is a rediscovery of a proof that first appeared in print in [Loomis, pp. 26-27]. The proof has been submitted by Sang Woo Ryoo, student, Carlisle High School, Carlisle, PA.

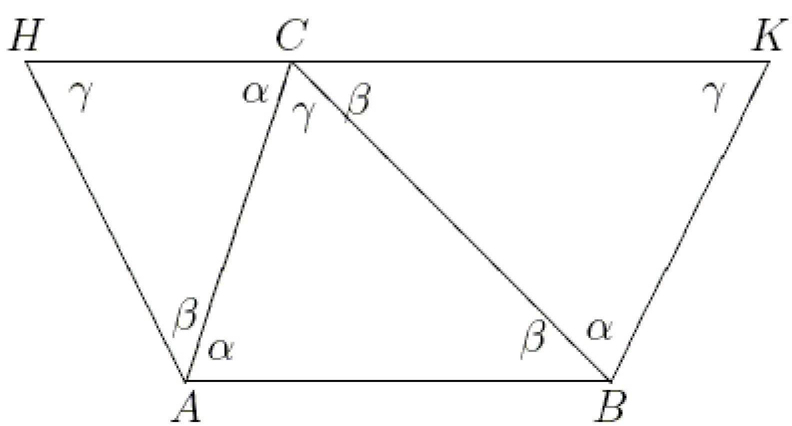
Loomis takes credit for the proof, although Monthly's editor traces its origin to a 1896 paper by B. F. Yanney and J. A. Calderhead (*Monthly*, v. 3, p. 65-67.)



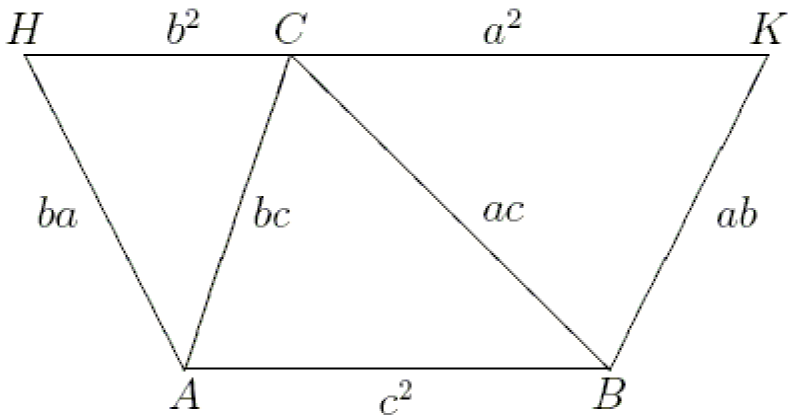
Draw AD, the angle bisector of angle A, and DE perpendicular to AB. Let, as usual, AB = c, BC = a, and AC = b. Let CD = DE = x. Then BD = a - x and BE = c - b. Triangles ABC and DBE are similar, leading to $x/(a - x) = b/c$, or $x = ab/(b + c)$. But also $(c - b)/x = a/b$, implying $c - b = ax/b = a^2/(b + c)$. Which leads to $(c - b)(c + b) = a^2$ and the Pythagorean identity.

Proof #83

This proof is a slight modification of the proof sent to me by Jan Stevens from Chalmers University of Technology and Göteborg University. The proof is actually of Dijkstra's generalization and is based on the extension of the construction in proof #41.



$\alpha + \beta > \gamma$



$a^2 + b^2 > c^2$.

The details can be found on a separate page.

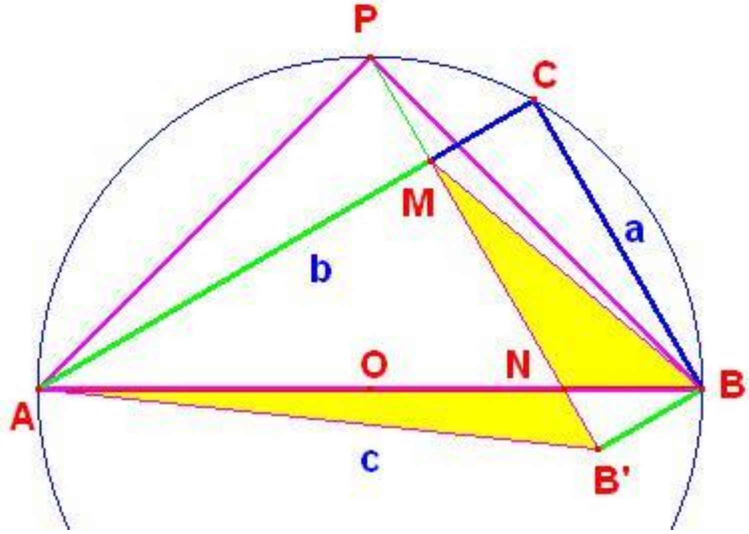
Proof #84

Elisha Loomis, myself and no doubt many others believed and still believe that no trigonometric proof of the Pythagorean theorem is possible. This belief stemmed from the assumption that any such proof would rely on the most fundamental of trigonometric identities $\sin^2\alpha + \cos^2\alpha = 1$ is nothing but a reformulation of the Pythagorean theorem proper. Now, Jason Zimba showed that the theorem can be derived from the subtraction formulas for *sine* and *cosine* without a recourse to $\sin^2\alpha + \cos^2\alpha = 1$. I happily admit to being in the wrong.

Jason Zimba's proof appears on a separate page.

Proof #85

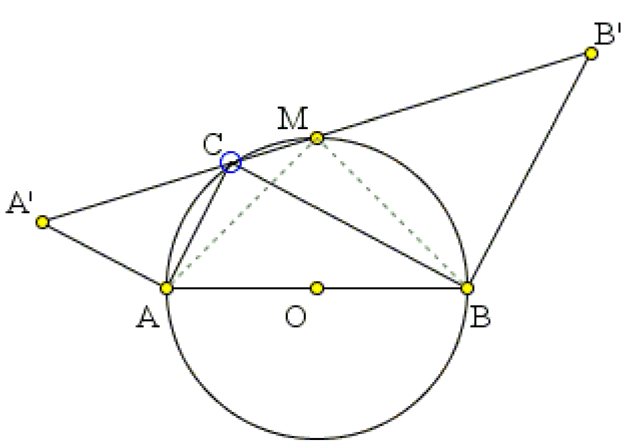
Bui Quang Tuan found a way to derive the Pythagorean Theorem from the Broken Chord Theorem.



For the details, see a separate page.

Proof #86

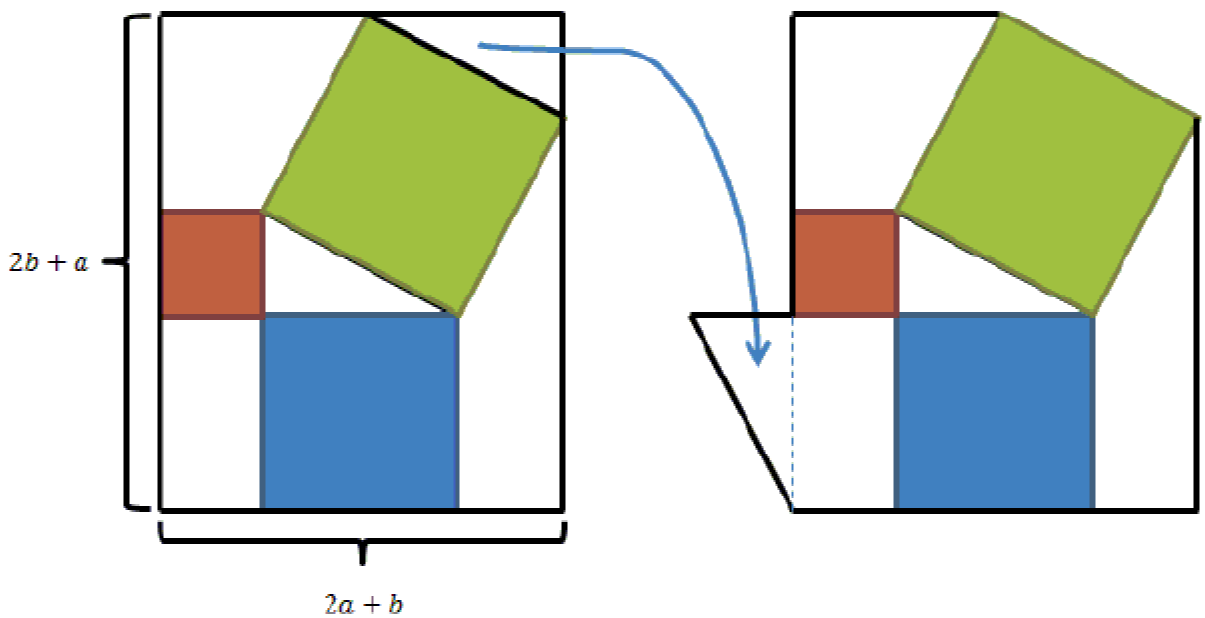
Bui Quang Tuan also showed a way to derive the Pythagorean Theorem from Bottema's Theorem.



For the details, see a separate page.

Proof #87

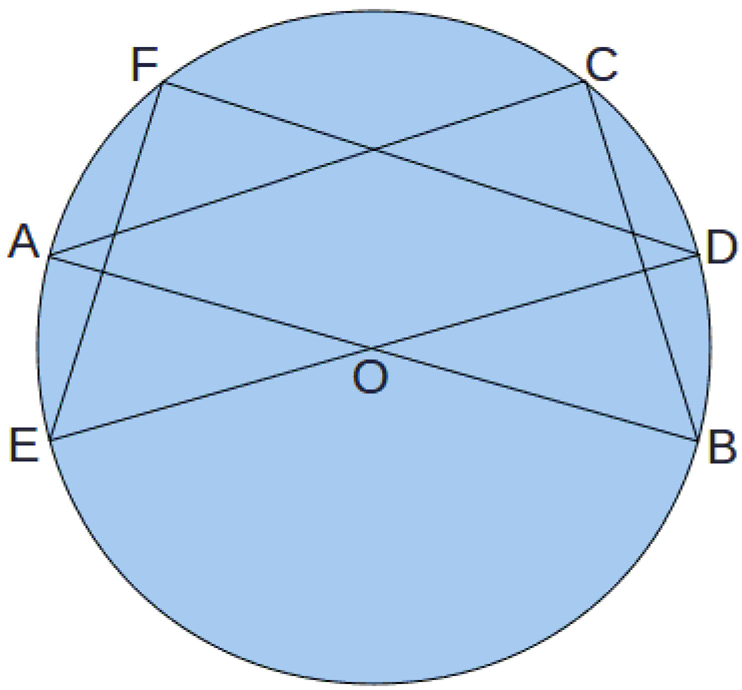
John Molokach came up with a proof of the Pythagorean theorem based on the following diagram:



If any proof deserves to be called algebraic this one does. For the details, see a separate page.

Proof #88

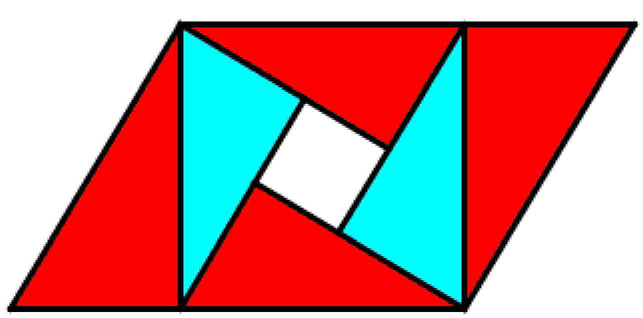
Stuart Anderson gave another derivation of the Pythagorean theorem from the Broken Chord Theorem. The proof is illustrated by the inscribed (and a little distorted) Star of David:



For the details, see a separate page. The reasoning is about the same as in Proof #79 but arrived at via the Broken Chord Theorem.

Proof #89

John Molokach, a devoted Pythagorean, found what he called a *Parallelogram* proof of the theorem. It is based on the following diagram:



For the details, see a separate page.

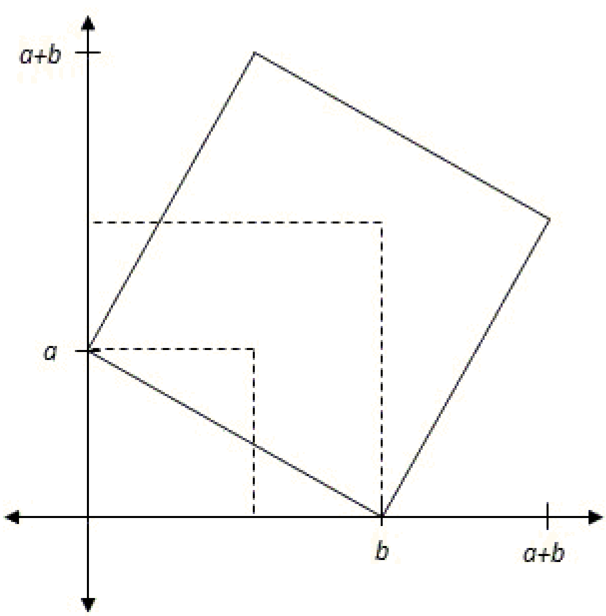
Proof #90

John has also committed an unspeakable heresy by devising a proof based on solving a differential equation. After a prolonged deliberation between Alexander Givental of Berkeley, Wayne Bishop of California State University, John and me, it was decided that the proof contains no vicious circle as was initially expected by every one.

For the details, see a separate page.

Proof #91

John Molokach also observed that the Pythagorean theorem follows from Gauss' Shoelace Formula:



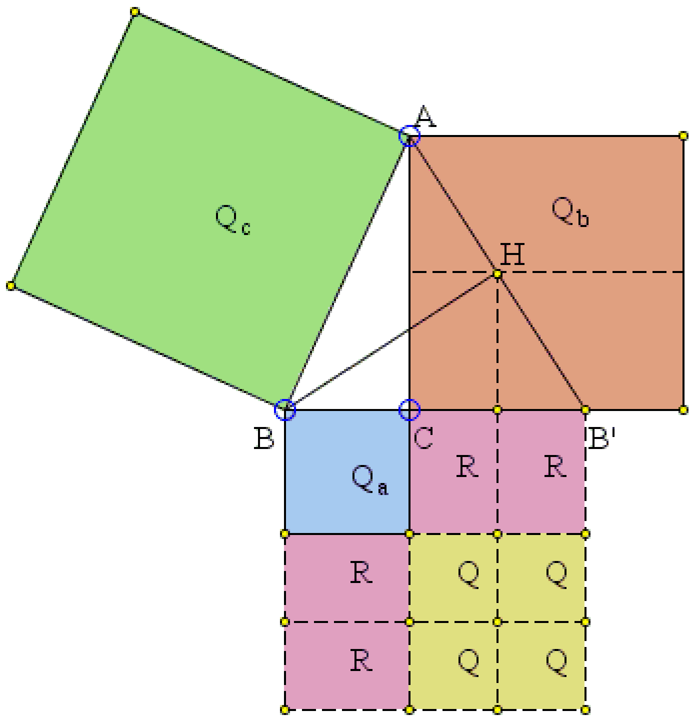
$$\begin{array}{c|cc|c} & b & 0 & \\ 0 & & & b^2 \\ & a+b & b & \\ ab & & & a^2+2ab+b^2 \\ & a & a+b & \\ 0 & & & a^2 \\ & 0 & a & \\ ab & & & 0 \\ & b & 0 & \end{array}$$

$$\frac{1}{2}[(2a^2+2ab+2b^2)-(2ab)]=a^2+b^2$$

For the details, see a separate page.

Proof #92

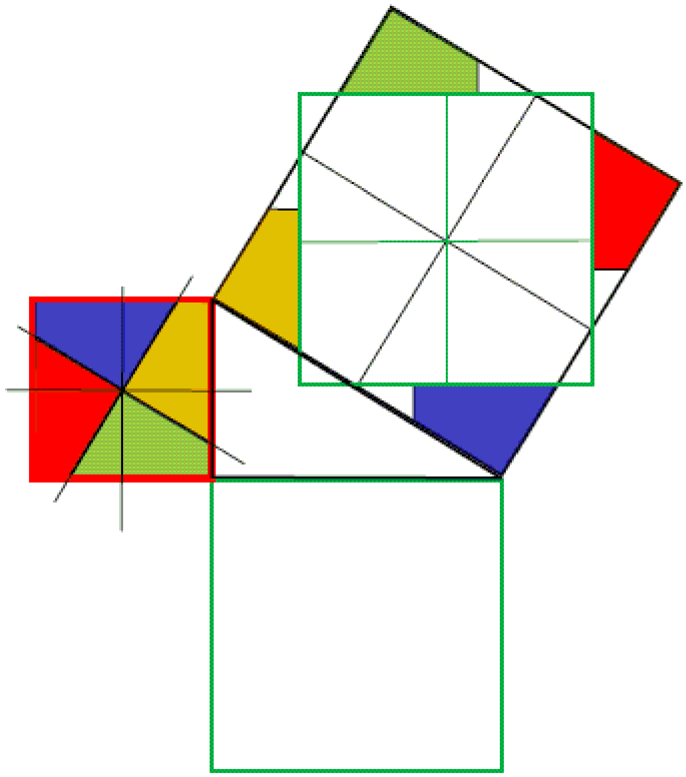
A proof due to Gaetano Speranza is based on the following diagram



For the details and an interactive illustration, see a separate page.

Proof #93

Giorgio Ferrarese from University of Torino, Italy, has observed that Perigal's proof - praised for the symmetry of the dissection of the square on the longer leg of a right triangle - admits further symmetric treatment. His proof is based on the following diagram



For the details, see a separate page.